L_p-theory for a class of viscoelastic fluids with and without a free surface

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Deutsche Zusammenfassung

Die vorliegende Dissertation beschäftigt sich mit nichtlinearen partiellen Differentialgleichungssystemen, die bestimmte Klassen von nicht-Newtonschen Fluiden modellieren. Es werden zum einen verallgemeinerte Newtonsche und zum anderen verallgemeinerte viskoelastische Fluide betrachtet. Die letzteren stellen dabei eine Verallgemeinerung des Oldroyd-B Modells dar. Die Wohlgestelltheit der um Anfangs- und Randbedingungen vervollständigten Systeme wird im Sinne der Theorie starker L_p-Lösungen untersucht.

Bevor genauer auf die Arbeit eingegangen wird, werden die untersuchten Modelle vorgestellt. Die Strömung eines inkompressiblen Fluids mit Dichte ρ, Geschwindigkeit u und Druck π in einem Gebiet $\Omega \subset \mathbb{R}^n$ auf einem Zeitintervall $[0, T]$ wird durch die Gleichungen

$$\rho\partial_t u + \rho u.\nabla u + \nabla\pi = \operatorname{Div} S + \rho f, \quad \operatorname{div} u = 0 \quad \text{in } (0, T) \times \Omega$$

beschrieben. Die Funktion f repräsentiert äußere Kräfte, die auf das Fluid wirken und der Tensor S Kräfte, die durch das Verformen des Fluids entstehen. Um das System mathematisch zu schließen, wird eine weitere Gleichung benötigt, die eine Relation zwischen der Bewegung des Fluids und des Tensors S herstellt. In dieser Arbeit werden zwei solcher Relationen in verschiedenen Kontexten untersucht: Zum einen wird das verallgemeinerte Newtonsche Gesetz

$$S = 2\alpha(|Eu|^2)Eu, \quad \alpha\colon [0, \infty) \to [0, \infty),$$

wobei $Eu = \frac{1}{2}(\nabla u + \nabla u^T)$ den symmetrischen Teil des Geschwindigkeitsgradienten bezeichnet, und zum anderen wird das verallgemeinerte viskoelastische Gesetz

$$S = 2\alpha(|Eu|^2)Eu + \mu(\tau), \quad \alpha\colon [0, \infty) \to [0, \infty), \quad \mu\colon \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n},$$

wobei τ durch die Transportgleichung

$$\partial_t\tau + u.\nabla\tau = g(\nabla u, \tau), \quad g\colon \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$$

gegeben ist, betrachtet.

Im ersten Teil der Arbeit wird das Modell für verallgemeinerte viskoelastische Fluide auf einem festen Gebiet Ω analysiert, dessen Rand $\partial\Omega = \Gamma_D \cup \Gamma_S$ sich in zwei disjunkte, offene und abgeschlossene Teilmengen Γ_D und Γ_S zerlegt. Dieses Modell wird mit "no-slip" Randbedingungen auf Γ_D ($u = 0$ auf Γ_D) und "perfect-slip" Randbedingungen auf Γ_S ($(u \cdot \nu, S\nu - (S\nu \cdot \nu)\nu = 0$ auf Γ_S) geschlossen. Für beschränkte Gebiete wird zeitlokale Existenz einer eindeutigen Lösung für Anfangswerte $(u_0, \tau_0) \in W_p^{2-\frac{2}{p}}(\Omega) \times H_p^1(\Omega)$ gezeigt, die natürlichen Kompatibilitätsbedingungen genügen. Unter der zusätzlichen Annahme, dass die Funktion $\alpha > 0$ konstant ist, überträgt sich dieses Resultat auf eine große Klasse unbeschränkter Gebiete (diese Klasse beinhaltet zum Beispiel Außenräume, Schichten, Halbräume und den Ganzraum) im Falle von “no-slip” Randbedingungen ($\Gamma_S = \emptyset$) und auf den Halbraum im Falle von “perfect-slip” Randbedingungen ($\Gamma_D = \emptyset$). Zum Beweis wird das Gleichungssystem als Fixpunktproblem formuliert. Im Falle beschränkter Gebiete kann dieses Fixpunktproblem unter Verwendung des Schauderschen Fixpunktsatzes gelöst und die Eindeutigkeit mittels Energieabschätzungen gezeigt werden. Da bei dieser Argumentation die Kompaktheit von Sobolev-Einbettungen wesentlich eingeht, ist diese Methode nicht direkt auf unbeschränkte Gebiete übertragbar. Da auch der Banachsche Fixpunktsatz nicht direkt anwendbar ist, wird im Falle von unbeschränkten Gebieten eine Variante dessen verwendet, bei der es hinreichend ist, die Kontraktion in einer schwächeren Topologie zu zeigen (siehe Proposition 1.13). Um diese Variante anzuwenden werden Abschätzungen der Lösung u zum Stokesproblem mit rechter Seite $\operatorname{Div} F$ der Form

$$\|u\|_{H_p^{\frac{1}{2}}(0,T;L_p(\Omega))} + \|u\|_{L_p(0,T;H_p^1(\Omega))} \leq C\|F\|_{L_p(0,T;L_p(\Omega))}$$

für einer großen Klasse von Gebieten gezeigt.

Der zweite Teil der Arbeit beschäftigt sich mit einem Zweiphasenproblem mit Oberflächenspannung, bei dem beide Phasen aus verallgemeinerten Newtonschen Flüssigkeiten bestehen. Beide Fluide sind durch eine Hyperfläche $\Gamma(t)$ getrennt, die zum Anfangszeitpunkt ($\Gamma_0 = \Gamma(0)$) als Graph einer Höhenfunktion gegeben ist ($\Gamma_0 = \operatorname{graph}(h_0)$). Die Bewegung des Fluids ist mittels der kinematischen Bedingung $V = u \cdot \nu$ an die Bewegung der Hyperfläche gekoppelt, wobei V die Geschwindigkeit von Γ in Normalenrichtung und ν die Normale an Γ bezeichnet, welche vom ersten zum zweiten Fluid zeigt. Es wird ein Modell betrachtet, bei dem die einzige Oberflächenkraft die Oberflächenspannung ist ($[\![S]\!]\nu = \sigma\kappa\nu$ auf $\Gamma(t)$, wobei $[\![\cdot]\!]$ den Sprung einer Größe auf $\Gamma(t)$, der Skalar σ die gegebene Oberflächenspannung und κ die Hauptkrümmung bezeichnet), und die Fluide an der Hyperfläche stetig sind ($[\![u]\!] = 0$ auf $\Gamma(t)$). Es wird die Existenz und Eindeutigkeit einer starken Lösung auf beliebigen endlichen Zeitintervallen für hinreichend kleine Anfangswerte $(u_0, h_0) \in W_p^{2-\frac{2}{p}}(\mathbb{R}^n \setminus \Gamma_0) \times W_p^{3-\frac{2}{p}}(\mathbb{R}^{n-1})$ bewiesen, die natürliche Kompatibilitätsbedingungen erfüllen. Weiter wird gezeigt, dass auch für positive Zeiten, die Hyperfläche $\Gamma(t)$ als Graph einer Höhenfunktion gegeben ist. Zum Beweis wird das Gleichungssystem durch die Hanzawa-Transformation auf ein festes Gebiet transformiert und als Fixpunktproblem formuliert. Unter Verwendung des Banachschen Fixpunktsatzes kann die Existenz einer eindeutigen Lösung gezeigt werden.

Anschließend wird noch einmal das Modell für verallgemeinerte viskoelastische Flüssigkeiten aus dem ersten Teil aufgegriffen. Unter Vernachlässigung der Oberflächenspannung wird ein zugehöriges freies Randwertproblem in Lagrange-Koordinaten analysiert. Es wird eine Situation betrachtet, in der das Gebiet $\Omega(t)$ zum Anfangszeitpunkt ($\Omega_0 = \Omega(0)$) kompakt berandet ist und sich der Rand $\partial\Omega(t) = \Gamma_F(t) \cup \Gamma_D$ in zwei disjunkte, offene und abgeschlossene Teilmengen $\Gamma_F(t)$ und Γ_D zerlegt. Der Teil Γ_D des Randes ist fixiert und es werden “no-slip” Randbedingungen Γ_D ($u = 0$ auf Γ_D) vorgeschrieben. Darüberhinaus ist der Teil $\Gamma_F(t)$ des Randes eine weitere Unbekannte, die, wie im zweiten Teil, durch die kinematische Bedingung $V = u \cdot \nu$ an die Bewegung des Fluids gekoppelt ist, wobei ν die äußere Normale an $\Omega(t)$ bezeichnet. Es wird ein Einphasenmodell ohne Oberflächenkräfte betrachtet ($S\nu = 0$ auf $\Gamma_F(t)$). Im Gegensatz zu den ersten beiden Teilen, wird das Modell in Lagrange-Koordinaten (und nicht in Euler-Koordinaten) untersucht. Ist u ein Geschwindigkeitsfeld in Euler-Koordinaten, dann ist die Transformation zwischen Euler-

Koordinaten x und Lagrange-Koordinaten ξ gegeben durch

$$x = X_u(t,\xi) := \xi + \int_0^t v(s,\xi)ds, \quad \xi \in \Omega_0, \quad t \in (0,T),$$

wobei $v(t,\xi) := u(t, X_u(t,\xi))$. Die Geometrie des freien Randes $\Gamma_F(t)$ ist nun durch die Relation $\Gamma_F(t) = \{X_u(t,\xi) \colon \xi \in \Gamma_{F,0}\}$ bestimmt. Der Übergang zu Lagrange-Koordinaten ergibt ein Gleichungssystem für die transformierten Unbekannten $(v,\theta,\eta)(t,\xi) := (u,\pi,\tau)(t, X_u(t,\xi))$ auf dem zeitunabhängigen Anfangsgebiet Ω_0. Für dieses wird die zeitlokale Existenz und Eindeutigkeit einer starken Lösung für Anfangswerte $(u_0,\tau_0) \in W_p^{2-\frac{2}{p}}(\Omega_0) \times H_p^1(\Omega_0)$ bewiesen, die natürlichen Kompatibilitätsbedingungen genügen. Zum Beweis wird das Gleichungssystem als Fixpunktproblem formuliert. Ein Vorteil des Lagrangschen Zugangs ist die einfache Form der linken Seite der Transportgleichung ($\partial_t \eta(t,\xi) = (\partial_t \tau + u.\nabla\tau)(t, X_u(t,\xi))$), denn diese ermöglicht, im Gegensatz zum ersten Teil, eine Anwendung des Banachschen Fixpunktsatzes zur Lösung des Fixpunktproblems.

Contents

Introduction

The aim of this thesis is to analyse a mathematical model of a generalized Newtonian fluid and a generalized viscoelastic fluid of differential type (a generalization of the Oldroyd-B model) on a fixed domain and with a free surface. By completing these models with appropriate initial and boundary conditions, we end up with a nonlinear system of partial differential equations. We investigate these on existence and uniqueness of strong L_p-solutions. However, before discussing this work more thoroughly, we present a physical motivation of the investigated models.

Physical Motivation

We consider the motion of an incompressible fluid with constant density ρ, occupying the region $\Omega \subset \mathbb{R}^n$ during the time $[0, T]$. The velocity of the fluid is denoted by u and the pressure of the fluid by π, and they are given by the equations

$$\left\{ \begin{array}{rcll} \rho\partial_t u + \rho u.\nabla u + \nabla\pi & = & \operatorname{Div} S + \rho f & \text{in } (0,T)\times\Omega, \\ \operatorname{div} u & = & 0 & \text{in } (0,T)\times\Omega. \end{array} \right. \tag{0.1}$$

Here, S is the extra stress tensor and f is related to the external forces. The first equation expresses the conservation of momentum and the second equation the conservation of mass. In response of being deformed, the fluid develops forces, which are described by S. To close the system mathematically, an addition constitutive equation, relating the extra stress to the motion of the fluid, is required.

One example of such a constitutive equation, is the Newtonian law. The relation of the extra stress to the motion is linear and is given by

$$S = 2\alpha E u, \quad \alpha > 0. \tag{0.2}$$

Here, α is the constant viscosity of the fluid and $Eu = \frac{1}{2}(\nabla u + \nabla u^T)$ is the symmetric part of the gradient. Adding an initial value and Dirichlet boundary conditions, we obtain the Navier-Stokes equation.

There are several ways to generalized the Newtonian law. A nonlinear variant, incorporating shear-thinning and shear-thickening effects, is the generalized Newtonian fluid model

$$S = 2\alpha(|Eu|^2)Eu, \quad \alpha\colon [0,\infty) \to [0,\infty). \tag{0.3}$$

Compared to the Newtonian model, the viscosity α is not constant, but a function depending on $|Eu|^2 = \sum_{j,k}(Eu)_{j,k}^2$. In this case, S is still an explicit function of the symmetric part of the gradient. A standard model of this kind is the power law, where a special form of the viscosity function α is assumed:

$$\alpha(s) = \alpha_0 + \alpha_1 s^{\frac{r-2}{2}}, \quad \alpha_0 \geq 0, \quad \alpha_1 > 0, \quad r \geq 1.$$

If $\alpha_0 > 0$ and $r = 2$, the power law model reduces to the Newtonian law.

In the case of Newtonian and generalized Newtonian fluid models, the extra part of the stress only depends on the current motion of the fluid. In viscoelastic fluid models, the history of the motion is also taken into account. An important viscoelastic fluid model is the Oldroyd-B model. Here, it is assumed that the extra stress

$$S = 2\alpha Eu + \tau, \quad \alpha > 0$$

decomposes in a viscous part $2\alpha Eu$ (of Newtonian type) and an elastic part τ, where the elastic part is given as the solution of the transport equation

$$\partial_t \tau + u.\nabla\tau - (\nabla u)\tau - \tau(\nabla u)^T + \beta\tau = 2\gamma Eu, \quad \beta, \gamma \geq 0.$$

A more general model of this kind is the generalized viscoelastic fluid model. Similar to the Oldroyd-B model, the extra part of the stress

$$S = 2\alpha(|Eu|^2)Eu + \mu(\tau), \quad \alpha\colon [0,\infty) \to [0,\infty), \quad \mu\colon \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$$

decomposes in a viscous part $2\alpha(|Eu|^2)Eu$ (of generalized Newtonian type) and an elastic part $\mu(\tau)$, where τ is defined as the solution of the transport equation

$$\partial_t \tau + u.\nabla\tau = g(\nabla u, \tau), \quad g\colon \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}.$$

This fluid model is a generalization of the Oldroyd-B model. Compared to the Oldroyd-B model, the viscous part $2\alpha Eu$ is not of Newtonian type, but of generalized Newtonian type and a more general form of the elastic part of the stress and of the transport equation is assumed. Examples of generalized viscoelastic fluid models are the generalized Oldroyd-B model, the White-Metzner model, as well as the Peterlin approximation (for an overview of viscoelastic fluid models, we refer the reader to Bird, Armstrong, and Hassager [BAH87] as well as Renardy [Ren00]). In all these mentioned models $\alpha > 0$ is constant and a special form of μ and g is assumed.

Overview of this thesis

In Chapter 1, we set up the notation and summarize some preliminary results. We review some standard facts on the linear theory, on function spaces, on trace and embedding theorems, and on Nemytskij operators. A detailed exposition on the transport equation and on the Fréchet differentiability of Nemytskij operators is given.

Chapter 2 deals with generalized viscoelastic fluid model on a fixed domain $\Omega \subset \mathbb{R}^n$. To complete the system, we add initial values for u and τ as well as boundary conditions. We investigate two different kind of boundary conditions on two disjoint boundary parts Γ_D and Γ_S (we assume that the boundary of the domain $\partial\Omega = \Gamma_D \cup \Gamma_S$ decomposes in two disjoint subsets Γ_D and Γ_S, which are open and closed in Ω). On the boundary part Γ_D, we impose Dirichlet boundary conditions on u ($u = 0$ on Γ_D) and on the part Γ_S, we require perfect slip boundary conditions ($(u \cdot \nu, S\nu - (S\nu \cdot \nu)\nu) = 0$ on Γ_S). If Ω is a bounded domain, we prove local existence of unique strong L_p-solutions, provided that the initial values belong to $(u_0, \tau_0) \in W_p^{2-\frac{2}{p}}(\Omega) \times H_p^1(\Omega)$ and satisfy natural compatibility conditions. In large class of unbounded domains (for example exterior domains with a smooth boundary, layers, half spaces, or the whole space) and under the additional assumption that the viscosity function is constant, we establish basically the same result, provided that $\Gamma_S = \emptyset$, as well as in the half space, provided that $\Gamma_D = \emptyset$.

In Chapter 3, we investigate a generalized Newtonian two-phase problem with surface tension. We analyse the motion of two different generalized Newtonian fluids (the fluids have different densities and viscosity functions), which are separated by an interface Γ. The interface is coupled to the motion of the fluids by the kinematic condition $V = u \cdot \nu$, where V is the normal velocity of the interface and ν is the normal, pointing from the first to the second fluid. Further, we impose the continuity of the velocities of the fluids on the interface ($[\![u]\!] = 0$ on Γ, where $[\![\cdot]\!]$ denotes the jump of a quantity on the boundary), and a special version of the momentum transmission condition, where the only surface force is the surface tension ($[\![S]\!]\nu = \sigma\kappa\nu$ on Γ, where σ denotes the surface tension and κ the mean curvature of the interface). We consider the situation, where initially the domain occupied by the fluids are close to half spaces and the interface, separating the fluids, is given as the graph of a hight function h_0 over $\mathbb{R}^{n-1}$ (graph$(h_0) = \Gamma_0$). We prove the existence of a unique strong L_p-solution on an arbitrary finite time interval, provided that the initial velocity $u_0 \in W_p^{2-\frac{2}{p}}(\mathbb{R}^n \setminus \Gamma_0)$ and $h_0 \in W_p^{3-\frac{2}{p}}(\mathbb{R}^{n-1})$ are sufficiently small and satisfy natural compatibility condition. Further, we prove that the interface Γ, separating the both fluids, is for all $t \in (0, T)$ given as the graph of a height function over $\mathbb{R}^{n-1}$.

Finally, in Chapter 4, we study a generalized viscoelastic free boundary problem without surface tension. In comparison to the both chapters before, we investigate the problem in Lagrangian coordinates instead of Eulerian coordinates. More precisely, we consider the viscoelastic fluid model on the domain $\Omega(t)$. We prescribe two different kinds of boundary conditions on two disjoint boundary parts Γ_D and $\Gamma_F(t)$ (we assume that the boundary of the initial domain $\partial\Omega_0 = \Gamma_D \cup \Gamma_{F,0}$ decomposes in two disjoint parts Γ_D and $\Gamma_{F,0}$, which are open and closed in $\partial\Omega_0$). The first boundary part Γ_D is fixed and we impose Dirichlet boundary conditions ($u = 0$ on Γ_D). The second boundary part $\Gamma_F(t)$ is unknown and is coupled the same way as in the third chapter to the motion of the fluid via $V = u \cdot \nu$. Since we consider a one phase problem and neglect the surface tension, we impose a special version of the momentum transmission condition without any surface force on Γ_F ($S\nu = 0$ on $\Gamma_F(t)$). The main result of this chapter is the local-in-time existence of strong L_p-solutions of the problem in the Lagrangian framework, provided that the initial domain Ω_0 admits a compact boundary, as well as the initial values belong to $(u_0, \tau_0) \in W_p^{2-\frac{2}{p}}(\Omega_0) \times H_p^1(\Omega_0)$ and satisfy natural compatibility conditions. The analysed model corresponds to the model in the second chapter, but in this chapter, we include the effect of a free surface. Compared to Chapter 3, we analyse a more general model, considering a one phase flow instead of a two phase flow and neglecting the effect of surface tension.

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Chapter 1

Preliminaries

In this chapter, we introduce the notation and preliminary results.

1.1 Notation

Most of the notation we use is standard. We always use a generic constant C and a generic function O with $O(t) \to 0$ for $t \to 0$, which may change from line to line, but is always independent of the free variables.

We denote by $\mathbb{N} = \{1, 2, 3, \dots\}$ the set of all positive integers and we put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The set of all real numbers is denoted by $\mathbb{R}$ and the set of all complex numbers by $\mathbb{C}$.

Let (K, d) be a metric space. We define the open R-neighbourhood of $x \in K$ by

$$B_K(x, R) := \{y \in K : d(x, y) < R\},$$

and by $\overline{B}_K(x, R) := \overline{B_K(x, R)}$ the closure of this neighbourhood.

Let X, Y be two Banach spaces. By $\mathcal{L}(X, Y)$, we define the Banach space, consisting of all linear and bounded maps $T : X \to Y$ and we put $\mathcal{L}(X) := \mathcal{L}(X, X)$. The dual space of X is denoted by X'. We use the notation $\rightharpoonup$ and $\overset{*}{\rightharpoonup}$ to denote the weak and weak-$*$ convergence respectively (that is, convergence with respect to the weak topology and weak-$*$ topology respectively).

Let $U \subset X$ be an open subset of X and $k \in \mathbb{N}_0$. The Banach space of all k-time continuously Fréchet differentiable functions $F : U \to Y$ is denoted by $C^k(U, Y)$ and we write for the Fréchet derivative $DF : Y \to \mathcal{L}(X, Y)$, provided that $F \in C^1(U, Y)$.

For a linear operator A in a Banach space X, we denote by $D(A)$ the domain, by $R(A)$ the range, by $N(A) = \{x \in D(A) : Ax = 0\}$ the kernel, and by $\rho(A)$ the resolvent set of this operator.

The real interpolation functor is denoted by $(\cdot, \cdot)_{\theta,q}$ and the complex interpolation functor by $[\cdot, \cdot]_\theta$, $0 < \theta < 1$, $1 \le q \le \infty$.

Let $n \in \mathbb{N}$, $n \ge 2$, $x, y \in \mathbb{R}^n$, and $A, B \in \mathbb{R}^{n \times n}$. The inner product is defined by

$$x \cdot y = \sum_{j=1}^{n} x_j y_j \quad \text{and} \quad A : B = \sum_{j,k=1}^{n} A_{j,k} B_{j,k}.$$

Moreover, we set $x = (x', x_n) \in \mathbb{R}^n$, with $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$, and we define

$$\mathbb{R}^n_\pm := \{x = (x', x_n) \in \mathbb{R}^n : \pm x_n > 0\} \quad \text{as well as} \quad \dot{\mathbb{R}}^n := \{x = (x', x_n) \in \mathbb{R}^n : x_n \neq 0\}.$$

Further, let $\Omega \subset \mathbb{R}^n$ be a domain with a C^1-boundary (boundary regularity is to be understood in the sense of Adams and Fournier [AF03, Definition 4.10]). The boundary of the domain is denoted by $\partial\Omega$ and its outer normal by ν. The normal part of a vector $x \in \partial\Omega$ is defined by $x_\nu := (x \cdot \nu)\nu$ and the tangential part by $x_{\tan} := x - x_\nu$. The divergence is defined by

$$\operatorname{div} f := \sum_{j=1}^n \partial_j f_j, \quad f\colon \Omega \to \mathbb{R}^n \quad \text{and} \quad \operatorname{Div} F := \big(\sum_{k=1}^n \partial_k F_{j,k}\big)_{j=1,\dots,n}, \quad F\colon \Omega \to \mathbb{R}^{n\times n},$$

the gradient is denoted by ∇ and the Laplace operator by Δ. The gradient and the Laplace operator operating on $\mathbb{R}^{n-1}$ is denoted by ∇' and Δ' respectively, more precisely, we put

$$\nabla' f := (\partial_1 f, \dots, \partial_{n-1} f)^T \quad \text{and} \quad \Delta' f := \sum_{j=1}^{n-1} \partial_j^2 f, \quad f\colon \mathbb{R}^{n-1} \to \mathbb{R}.$$

Next, we introduce basic function spaces. Fix $d \in \mathbb{N}$, $k \in \mathbb{N}_0$, $1 \le p, q \le \infty$, and $s \ge 0$. Let X be a Banach space and G be an open subset of $\mathbb{R}^d$. We denote by $C^k(G,X)$ the set of all k-times continuous differentiable functions $f\colon G \to X$ and by $C_c^k(G,X)$ the subset of $C^k(G,X)$, where all functions have a compact support. Further, we introduce $BUC^k(G,X)$, the space of all bounded and uniformly continuous functions with bounded and uniformly continuous derivatives up to order k. We write $L_p(G,X)$ for the usual X-valued Lebesgue space, $H_p^s(G,X)$ for the usual X-valued Bessel potential space, $W_p^s(G,X)$ for the usual X-valued Sobolev-Slobodeckii space, and $B_{p,q}^s(G,X)$ for the usual X-valued Besov space. The corresponding homogeneous spaces are denoted by $\widehat{H}_p^s(G,X)$, $\widehat{W}_p^s(G,X)$, $\widehat{B}_{p,q}^s(G,X)$. For simplification, we write $C(G,X) = C^0(G,X)$ as well as $BUC(G,X) = BUC^0(G,X)$. For $s \in (0,\infty) \setminus \mathbb{N}$, an equivalent norm in $W_p^s(G,X)$ is defined by

$$\|f\|_{W_p^s(G,X)} := \|f\|_{W_p^{[s]}(G,X)} + [f]_{W_p^s(G,X)},$$

where $[s]$ is the largest integer smaller than s and

$$[f]_{W_p^s(G,X)} := \sum_{|\alpha|=[s]} \Big(\int_{G\times G} \frac{\|\partial^\alpha f(x) - \partial^\alpha f(y)\|_X^p}{|x-y|^{d+(s-[s])p}} dxdy \Big)^{\frac{1}{p}}.$$

For $T > 0$, we write

$$\mathcal{J}_p^s(0,T;\mathcal{K}_q^r(G)) := \mathcal{J}_p^s((0,T),\mathcal{K}_q^r(G)), \quad \mathcal{J},\mathcal{K} \in \{H,W\}, \quad 1 \le r \le \infty,$$

as well as

$$\|f\|_{G,p} := \|f\|_{L_p(G)}, \quad f \in L_p(G) \quad \text{and} \quad \|f\|_{T,G,p,q} := \|f\|_{L_p(0,T;L_q(G))}, \quad f \in L_p(0,T;L_q(G))$$

to shorten notation. We identify $L_p(0,T;L_p(G))$ with $L_p((0,T)\times G)$, provided that $1 < p < \infty$. Further, we introduce the dual pairing

$$(f|g)_G := \int_G fg dx, \quad (f,g) \in (L_p(G) \times L_{p'}(G)),$$

and

$$(f|g)_{T,G} := \int_T \int_G fg dx, \quad (f,g) \in L_p(0,T;L_q(G)) \times L_{p'}(0,T;L_{q'}(G)),$$

where $1 \le p', q' \le \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. If M is a closed and compact d-dimensional C^m-manifold, $m \ge s$ and $m \ge k$, we use analogously the notation $C^k(M, X)$, $BUC^k(M, X)$, $L_p(M, X)$, $H^s_p(M, X)$, $W^s_p(M, X)$, and $B^s_{p,q}(M, X)$ and we use the same abbreviations.

Let Ω be a domain with a uniform C^2-boundary and let $\Gamma \subset \partial\Omega$ be a open and closed subset of $\partial\Omega$. The trace operator on the boundary part Γ is denoted by γ_Γ. For $1 < p < \infty$, we define the space

$$ {}_0H^1_{p,\Gamma}(\Omega) := \{f \in H^1_p(\Omega) \colon \gamma_\Gamma f = 0\} \quad \text{and its dual} \quad {}^0H^{-1}_{p,\Gamma}(\Omega) = ({}_0H^1_{p',\Gamma}(\Omega))', $$

where $1 < p' < \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$. If $\Gamma = \partial\Omega$, we write $H^1_{p,0}(\Omega) = {}_0H^1_{p,\partial\Omega}(\Omega)$ as well as $H^{-1}_p(\Omega) := {}^0H^{-1}_{p,\partial\Omega}(\Omega)$. Further, we put $H^{-1}_{p,0}(\Omega) := (H^1_{p'}(\Omega))'$. In an analogue way, we define ${}_0\widehat{H}^1_{p,\Gamma}(\Omega)$, ${}^0\widehat{H}^{-1}_{p,\Gamma}(\Omega)$, $\widehat{H}^{-1}_p(\Omega)$, and $\widehat{H}^{-1}_{p,0}(\Omega)$.

1.2 Linear Theory

In the most proofs of our main results, we solve a nonlinear problem by linearizing the problem and applying a fixed point argument. In order to apply a fixed point argument, it is important to have a proper understanding of the associated linearization. This section is a brief summary on some results on linear problems, which are relevant to this thesis.

1.2.1 Maximal regularity, bounded imaginary powers, and $\mathcal{H}^\infty$-calculus

In this subsection, we briefly discuss abstract properties of linear operators. For a thorough treatment, we refer the reader to Denk, Hieber, and Prüß [DHP03]. First, we recall the definition of a sectorial operator. For this purpose, we define the open sector $\Sigma_\theta \subset \mathbb{C}$ with opening angle 2θ by

$$ \Sigma_\theta := \{\lambda \in \mathbb{C} \colon |\arg \lambda| < \theta\}, \quad \theta \in (0, \pi]. $$

Definition 1.1. Let X be a Banach space and A a linear operator in X. The operator A is called sectorial, if A is closed, injective, $\overline{D(A)} = \overline{R(A)} = X$, and there exists an angle $\theta \in (0, \pi]$ and a constant $C > 0$, such that $\rho(-A) \supset \Sigma_\theta$ and

$$ \|\lambda(\lambda + A)^{-1}\|_{\mathcal{L}(X)} \le C, \quad \lambda \in \Sigma_\theta. $$

If A is a sectorial operator in X, we define the spectral angle by

$$ \phi_A := \inf_{\phi \in (0,\pi]} \{\rho(-A) \supset \Sigma_{\pi-\phi}, \sup_{\lambda \in \Sigma_\phi} \|\lambda(\lambda + A)^{-1}\|_{\mathcal{L}(X)} < \infty\}. $$

For a sectorial operator A in a Banach space X, one can develop a functional calculus. We denote by $\mathcal{H}^\infty(\Sigma_\phi)$, $\phi \in (0, \pi]$, the set of all holomorphic and bounded functions $f \colon \Sigma_\phi \to \mathbb{C}$ and we define

$$ \mathcal{H}_0(\Sigma_\phi) = \{f \in \mathcal{H}^\infty(\Sigma_\phi) \colon \text{there exist } C, \varepsilon > 0 \text{ with } |f(\lambda)| \le \Big(\frac{\lambda}{(\lambda+1)^2}\Big)^\varepsilon < C,\ \lambda \in \Sigma_\phi\}. $$

It is worth pointing out, that functions in $\mathcal{H}_0(\Sigma_\phi)$ decay at zero and infinity. For $\phi \in (\phi_A, \pi)$ and $\psi \in (\phi_A, \phi)$, the integral

$$ f(A) = \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda - A)^{-1} d\lambda, \quad f \in \mathcal{H}_0(\Sigma_\phi), $$

where Γ denotes the path surrounding Σ_ψ counterclockwise, defines via $\Phi_A(f) = f(A)$ a functional calculus $\Phi_A \colon \mathcal{H}_0(\Sigma_\phi) \to \mathcal{L}(X)$, which is a bounded algebra homomorphism (see [DHP03, Theorem 1.7]). This functional calculus can be extended to holomorhic functions $f\colon \Sigma_\phi \to \mathbb{C}$, which grow at most polynomially at zero and infinity (see [DHP03, Theorem 2.1]). For an unbounded function, the operator $f(A)$ is a densely defined and, in general, unbounded operator.

Definition 1.2. Let X be an Banach space and A a sectorial operator in X.

(a) A admits bounded imaginary powers, if $A^{is} \in \mathcal{L}(X)$ for all $s \in \mathbb{R}$ (A^{is} is defined via the extended functional calculus) and there exists a constant $C > 0$, such that

$$\|A^{is}\|_{\mathcal{L}(X)} \leq C, \quad s \in [-1, 1].$$

If A admits bounded imaginary powers, we define the power angle θ_A by

$$\theta_A := \limsup_{|s|\to\infty} \frac{1}{|s|} \log \|A^{is}\|_{\mathcal{L}(X)}.$$

(b) A admits a bounded $\mathcal{H}^\infty$-calculus, if there exists $\phi \in (\phi_A, \pi)$ and a constant C, such that

$$\|f(A)\|_{\mathcal{L}(X)} \leq C\|f\|_{\Sigma_\phi,\infty}, \quad f \in \mathcal{H}_0(\Sigma_\phi).$$

If A admits a bounded $\mathcal{H}^\infty$-calculus, we define the $\mathcal{H}^\infty$-angle ϕ_A^∞ by

$$\phi_A^\infty = \inf_{\phi\in(\phi_A,\pi)} \{\text{there exists a } C > 0, \text{ such that } \|f(A)\|_{\mathcal{L}(X)} \leq C\|f\|_{\Sigma_\phi,\infty},\ f \in \mathcal{H}_0(\Sigma_\phi)\}.$$

Remark 1.3. Let A be a sectorial operator in a Banach space X.

(a) If A admits a bounded $\mathcal{H}^\infty$-calculus and $\phi_A^\infty < \phi \leq \pi$, then the functional calculus for A on $\mathcal{H}_0(\Sigma_\phi)$ extends uniquely to $\mathcal{H}^\infty(\Sigma_\phi)$.

(b) If A admits bounded imaginary powers, then

$$\phi_A \leq \theta_A < \infty.$$

If A admits a bounded $\mathcal{H}^\infty$-calculus, then A admits bounded imaginary powers and

$$\phi_A \leq \theta_A \leq \phi_A^\infty.$$

For a proof, we refer the reader to [DHP03, (2.15) and (2.16)].

If an operator admits bounded imaginary powers, a characterisation of the domain of its factional powers can be established.

Proposition 1.4. *Let A be an operator in a Banach space X admitting bounded imaginary powers. Then*

$$D(A^\alpha) = [X, D(A)]_\alpha, \quad \alpha \in (0, 1).$$

For a proof, we refer the reader to [Tri78, Theorem 1.15.3].

Next, we define maximal regularity of an abstract Cauchy problem. Basically, we say that a linear operator admits maximal regularity, if the solution operator to the corresponding abstract Cauchy problem is a diffeomorphism between the data and the solution space.

Definition 1.5. Fix $0 < T < \infty$ and $1 < p < \infty$. Let X be a Banach space and A be a closed, linear, and unbounded operator in X with dense domain $D(A)$. We say A admits maximal L_p-regularity on $(0,T)$, if the map

$$H_p^1(0,T;X) \cap L_p(0,T;D(A)) \to L_p(0,T;X) \times (X,D(A))_{1-\frac{1}{p},p}, \quad u \mapsto (u' + Au, u(0))$$

is a linear and bounded diffeomorphism.

Remark 1.6. Fix $n \in \mathbb{N}$, $n \geq 2$, and $1 < p,q < \infty$. Let $\Omega \subset \mathbb{R}^n$ be a domain and let $X = L_q(\Omega)$ or let X be a closed subspace of $L_q(\Omega)$. Assume that A is a linear operator in X and $\lambda_0 \in \mathbb{C}$, such that $\lambda_0 + A$ admits bounded imaginary powers with power angle $\theta_{\lambda_0+A} < \frac{\pi}{2}$. Then, A admits maximal L_p-regularity. This result was established by Dore and Venni [DV87, Theorem 3.2], even for a more general class of Banach spaces X.

1.2.2 Stokes operator

Let Ω be a domain with a uniform C^2-boundary, which will be specified later. In the case of a Newtonian fluid, the extra part of the stress is given by $S = \alpha Eu$, where $\alpha > 0$ is a positive constant (see (0.2)). Without loss of generality, we can assume that $\alpha = \rho = 1$. Plugging this stress into the equation of motion (0.1) and adding a Dirichlet boundary condition and an initial value for the velocity field, we obtain the Navier-Stokes equation

$$\begin{cases} \partial_t u - \Delta u + u.\nabla u + \nabla\pi &= f \qquad \text{in } (0,T)\times\Omega, \\ \operatorname{div} u &= 0 \qquad \text{in } (0,T)\times\Omega, \\ u &= 0 \qquad \text{on } (0,T)\times\partial\Omega, \\ u(0) &= u_0 \qquad \text{in } \Omega. \end{cases}$$

The associated linearization is the Stokes problem

$$\begin{cases} \partial_t u - \Delta u + \nabla\pi &= f \qquad \text{in } (0,T)\times\Omega, \\ \operatorname{div} u &= 0 \qquad \text{in } (0,T)\times\Omega, \\ u &= 0 \qquad \text{on } (0,T)\times\partial\Omega, \\ u(0) &= u_0 \qquad \text{in } \Omega. \end{cases} \tag{1.1}$$

Next, we introduce the Helmholtz projection, in order to write the Stokes problem in the form of an abstract Cauchy problem.

Fix $1 < q < \infty$. By $C^\infty_{c,\sigma}(\Omega)$, we denote the divergence free test functions and we define

$$G_q(\Omega) := \nabla\widehat{H}^1_q(\Omega) \quad \text{and} \quad L_{q,\sigma}(\Omega) := \overline{C^\infty_{c,\sigma}(\Omega)}^{\|\cdot\|_{\Omega,q}}.$$

We say the Helmholtz decomposition exists for $L_q(\Omega)$, if

$$L_q(\Omega) = L_{q,\sigma}(\Omega) \oplus G_q(\Omega),$$

where $\oplus$ denotes the direct sum. In this case, there exists a unique projection $P_q\colon L_q(\Omega) \to L_{p,\sigma}(\Omega)$ with $N(P_q) = G_q(\Omega)$. This projection is called Helmholtz projection. The Helmholtz decomposition exists for example for bounded and exteriors domains with a smooth boundary, half spaces and the whole space. If the Helmholtz decomposition exists, we define the Stokes operator

$$A_q\colon D(A_q) \subset L_{q,\sigma}(\Omega) \to L_{q,\sigma}(\Omega), \quad u \mapsto -P_q \Delta u,$$

with

$$D(A_q) = H^2_q(\Omega) \cap H^1_{q,0}(\Omega) \cap L_{q,\sigma}(\Omega).$$

In this situation, we rewrite (1.1) equivalently in the form

$$u' + A_q u = P_q f \quad \text{in } (0,T), \quad u(0) = u_0. \tag{1.2}$$

More precisely, $u \in H^1_p(0,T; L_{q,\sigma}(\Omega)) \cap L_p(0,T; D(A_q))$, $1 < p, q < \infty$, solves (1.1) if and only if u solves (1.2) Maximal regularity results of the Stokes operator go back to Solonnikov [Sol77a]. Recently, Geißert, Heck, Hieber, and Sawada [GHHS12] proved, that for $1 < p, q < \infty$ and $0 < T < \infty$ the Stokes operator A_q admits maximal L_p-regularity on $(0,T)$ for a large class of domains. The fact that the Stokes operator on bounded domains admits bounded imaginary powers, was established by Giga [Gig85]. Noll and Saal [NS03] proved the existence of a bounded $\mathcal{H}^\infty$-calculus for the Stokes operator in domains with a compact boundary. Recently, Abels and Terasawa [AT09] extended the result on the bounded $\mathcal{H}^\infty$-calculus to a wide class of domains, where they actually considered a non-constant viscosity.

Next, following the argumentation of Amann [Ama00, Theorem 3.4], we analyse the Stokes scale.

Lemma 1.7. *Fix $1 < q, q' < \infty$ with $\frac{1}{q} + \frac{1}{q'} = 1$. Let $r \in \{q, q'\}$ and let $\Omega \subset \mathbb{R}^n$ be a uniform C^2-domain, such that the Helmholtz decomposition exists for $L_r(\Omega)$ and the Stokes operator A_r generates an analytic semigroup. Then,*

$$\begin{aligned}&(L_{q,\sigma}(\Omega), D(A_q))_{\theta,p} \\ &\quad = \begin{cases} B^{2\theta}_{q,p}(\Omega) \cap L_{q,\sigma}(\Omega) & 0 \le \theta < \frac{1}{2q} \\ \{u \in B^{2\theta}_{q,p}(\Omega)\colon u|_{\partial\Omega} = 0\} \cap L_{q,\sigma}(\Omega) & \frac{1}{2q} < \theta \le 1 \end{cases}, \quad \theta \in (0,1) \setminus \{\frac{1}{2q}\}, \quad p \in (1,\infty),\end{aligned}$$

and

$$[L_{q,\sigma}(\Omega), D(A_q)]_\theta = \begin{cases} H^{2\theta}_q(\Omega) \cap L_{q,\sigma}(\Omega) & 0 \le \theta < \frac{1}{2q} \\ \{u \in H^{2\theta}_q(\Omega)\colon u|_{\partial\Omega} = 0\} \cap L_{q,\sigma}(\Omega) & \frac{1}{2q} < \theta \le 1 \end{cases}, \quad \theta \in (0,1) \setminus \{\frac{1}{2q}\}.$$

Proof. First, choose $\lambda_0 > 0$ with $\lambda_0 \in \rho(-A_q)$. We denote the Dirichlet Laplace operator with domain

$$D(\Delta_D) = H^2_q(\Omega) \cap H^1_{q,0}(\Omega)$$

by Δ_D. First, we show that the adjoint $(A_q)'$ corresponds with $A_{q'}$: It holds that

$$(A_q f|g)_\Omega = -(\Delta_D f|g)_\Omega = -(f|\Delta_D g)_\Omega = (f|A_{q'} g)_\Omega, \quad (f,g) \in D(A_q) \times D(A_{q'}),$$

which implies $(A_q)' \supset A_{q'}$. $(A_q)'$ is densely defined and therefore generates an analytic semigroup. Since $A_{q'}$ also generates an analytic semigroup, the resolvents $\rho((A_q)')$ and $\rho(A_{q'})$ have a nonempty intersection, and hence $(A_q)' = A_{q'}$.

We define

$$Q := (\lambda_0 + A_q)^{-1} P_q(\lambda_0 - \Delta_D) f, \quad f \in D(\Delta_D).$$

The operator Q is a projection onto $D(A_q)$, since

$$Qf = (\lambda_0 + A_q)^{-1} P_q(\lambda_0 - \Delta_D) f = f, \quad f \in D(A_q),$$

where we used $P_q(\lambda_0 - \Delta_D)f = (\lambda_0 + A_q)f$, $f \in D(A_q)$, and hence

$$Q^2 f = Qf, \quad f \in D(\Delta_D).$$

Further, for $(f, g) \in D(\Delta_D) \times L_{q'}(\Omega)$, we deduce that

$$(Qf|g)_\Omega = ((\lambda_0 + A_q)^{-1} P_q(\lambda_0 - \Delta_D) f|g)_\Omega = (f|(\lambda_0 - \Delta_D)(\lambda_0 + A_{q'})^{-1} P_{q'} g)_\Omega,$$

and therefore

$$\|Qf\|_{\Omega,q} \leq C\|f\|_{\Omega,q}, \quad f \in D(\Delta_D).$$

Since $D(\Delta_D)$ is dense in $L_q(\Omega)$, we extend Q to a bounded projection $Q \in \mathcal{L}(L_q(\Omega))$ and we have $Q(L_q(\Omega)) \subset L_{q,\sigma}(\Omega)$. By the continuity of Q and the density of $D(A_q)$ in $L_{q,\sigma}(\Omega)$, we deduce that $Qf = f$, $f \in L_{q,\sigma}(\Omega)$. In summary we have a projection Q, with

$$Q \in \mathcal{L}(L_q(\Omega)) \quad \text{with} \quad Q(L_q(\Omega)) = L_{q,\sigma}(\Omega) \quad \text{and} \quad Q \in \mathcal{L}(D(\Delta_D)) \quad \text{with} \quad Q(D(\Delta_D)) = D(A_q).$$

By Triebel [Tri78, Theorem 1.17.1], it follows that

$$\mathfrak{F}(L_{q,\sigma}(\Omega), D(A_q)) = \mathfrak{F}(L_q(\Omega), D(\Delta_D)) \cap L_{q,\sigma}(\Omega),$$

where $\mathfrak{F}$ is an arbitrary interpolation functor. By Amann [Ama00, Theorem 2.2], it holds that

$$(L_q(\Omega), D(\Delta_D))_{\theta,p} = \begin{cases} B^{2\theta}_{q,p}(\Omega) & 0 \leq \theta < \frac{1}{2q}, \\ \{u \in B^{2\theta}_{q,p}(\Omega) \colon u|_{\partial\Omega} = 0\} & \frac{1}{2q} < \theta \leq 1, \end{cases} \quad \theta \in (0,1) \setminus \{\frac{1}{2q}\}, \quad p \in (1, \infty),$$

and

$$[L_q(\Omega), D(\Delta_D)]_\theta = \begin{cases} H^{2\theta}_q(\Omega) & 0 \leq \theta < \frac{1}{2q}, \\ \{u \in H^{2\theta}_q(\Omega) \colon u|_{\partial\Omega} = 0\} & \frac{1}{2q} < \theta \leq 1, \end{cases} \quad \theta \in (0,1) \setminus \{\frac{1}{2q}\}.$$

This completes the proof. $\square$

These spaces can be characterized more precisely in the case that Ω is a bounded C^2-domain, an exterior C^2-domain with $n \geq 3$, or a half space. We have (see [Ama00, Remark 3.7] and the references therein)

$$(1.3) \quad (L_{q,\sigma}(\Omega), D(A_q))_{\theta,p} = \{u \in B^{2\theta}_{q,p}(\Omega) \colon \operatorname{div} u = 0,\, u|_{\partial\Omega} = 0\}, \quad 1 < p, q < \infty, \quad \frac{1}{2q} < \theta < 1,$$

and

$$[L_{q,\sigma}(\Omega), D(A_q)]_\theta = \{u \in H^{2\theta}_q(\Omega) \colon \operatorname{div} u = 0,\, u|_{\partial\Omega} = 0\}, \quad 1 < q < \infty, \quad \frac{1}{2q} < \theta < 1.$$

1.2.3 Solvability of the generalized Stokes equation on fixed domains

One aim of this work is to bring generalized Newtonian fluids and viscoelastic fluids together. Bothe and Prüß [BP07] proved the local-in-time existence of a unique strong solution of the generalized Navier-Stokes equation. A basic tool in their proof is the unique solvability of the associated linearization, the so-called generalized Stokes problem as well as estimates of the solution. We introduce the generalized Stokes problem and recall their result.

Let $n \in \mathbb{N}$, $n \geq 2$, $n+2 < p < \infty$, $\alpha \in C^{1,1}([0,\infty))$, and $\Omega \subset \mathbb{R}^n$ be a domain with a compact $C^{2,1}$-boundary, such that the boundary $\partial\Omega = \Gamma_D \cup \Gamma_S \cup \Gamma_N$ decomposes in three disjoint subsets Γ_D, Γ_S, and Γ_N, which are open and closed in $\partial\Omega$. The outer normal on the boundary is denoted by ν. In the generalized Navier-Stokes equation, the divergence of the extra part of the stress $S = \alpha(|Eu|^2)Eu$ appears (see (0.1) and (0.3)). We compute this divergence to the result

$$\begin{aligned}
(\operatorname{Div} S)_j &= (\operatorname{Div} 2\alpha(|Eu|^2)Eu)_j \\
&= \sum_{l=1}^{n} \partial_l (2\alpha(|Eu|^2)(Eu)_{j,l}) \\
&= \sum_{l=1}^{n} 2\alpha(|Eu|^2)(\partial_l Eu)_{j,l} + 4\alpha'(|Eu|^2)(Eu : \partial_l Eu)(Eu)_{j,l} \\
&= \sum_{l=1}^{n} \alpha(|Eu|^2)(\partial_l^2 u_j + \partial_l \partial_j u_l) + \sum_{k,l,m=1}^{n} 2\alpha'(|Eu|^2)(Eu)_{k,m}(Eu)_{j,l}(\partial_l\partial_k u_m + \partial_l \partial_m u_k) \\
&= \sum_{l=1}^{n} \alpha(|Eu|^2)(\partial_l^2 u_j + \partial_l \partial_j u_l) + \sum_{k,l,m=1}^{n} 4\alpha'(|Eu|^2)(Eu)_{k,m}(Eu)_{j,l}\partial_l \partial_m u_k \\
&= \sum_{k,l,m=1}^{n} \mathcal{A}_{j,k}^{l,m}(Eu)\partial_l\partial_m u_k, \quad u \in H_p^2(\Omega), \quad j = 1, \ldots, n,
\end{aligned}$$

where we used $Eu = \frac{1}{2}(\nabla u + (\nabla u)^T)$ and the definition of the coefficients

$$\mathcal{A}_{j,k}^{l,m}(Eu) := \alpha(|Eu|^2)(\delta_{l,m}\delta_{j,k} + \delta_{j,m}\delta_{k,l}) + 4\alpha'(|Eu|^2)(Eu)_{j,l}(Eu)_{k,m}, \quad j,k,l,m = 1, \ldots, n.$$

This motivates the definition of the quasilinear second order operator

$$\mathcal{A}(Eu_*)u := -\big(\sum_{k,l,m=1}^{n} \mathcal{A}_{j,k}^{l,m}(Eu_*)\partial_l\partial_m u_k\big)_{j=1,\ldots,n}, \quad u, u_* \in H_p^2(\Omega).$$

It is worth pointing out, that

$$\mathcal{A}(Eu)u = -\operatorname{Div} 2\alpha(|Eu|^2)Eu \quad \text{and} \quad \mathcal{A}(0)u = -\alpha(0)\Delta u - \alpha(0)\nabla \operatorname{div} u, \quad u \in H_p^2(\Omega). \tag{1.4}$$

Let from now on $u_* \in H_p^2(\Omega)$. We define additionally two boundary operators; the Neumann boundary operator

$$\mathcal{B}_{\mathcal{N}}(Eu_*)(u,\pi) := \big(\sum_{k,l,m=1}^{n} \mathcal{A}_{j,k}^{l,m}(Eu_*)\nu_l\partial_m u_k\big)_{j=1,\ldots,n} - \pi\nu, \quad u \in H_p^2(\Omega), \quad \pi \in W^{1-\frac{1}{p}}(\Gamma_N),$$

and the perfect slip boundary operator

$$\mathcal{B}_\mathcal{S}(Eu_*)u := \big[\big(\sum_{k,l,m}^{n} \mathcal{A}_{j,k}^{l,m}(Eu_*)\nu_l\partial_m u_k\big)_{j=1,\dots,n}\big]_{\tan}, \quad u \in H_p^2(\Omega).$$

A different representation of the boundary operators will be used in Chapter 2 and Chapter 4. To obtain this representation, we compute the sum appearing in both boundary conditions, inserting the explicit representation of $\mathcal{A}_{j,k}^{l,m}(Eu_*)$. Using $Eu_* = (Eu_*)^T$, we compute

$$\begin{aligned}
&\Big(\sum_{k,l,m=1}^{n} \mathcal{A}_{j,k}^{l,m}(Eu_*)\nu_l\partial_m u_k\Big)_{j=1,\dots,n}\\
&= \Big(\sum_{k,l,m=1}^{n} \big(\alpha(|Eu_*|^2)(\delta_{l,m}\delta_{j,k} + \delta_{j,m}\delta_{k,l}) + 4\alpha'(|Eu_*|^2)(Eu_*)_{j,l}(Eu_*)_{k,m}\big)\nu_l\partial_m u_k\Big)_{j=1,\dots,n}\\
&= \Big(\sum_{l=1}^{n} 2\alpha(|Eu_*|^2)(Eu)_{j,l}\nu_l + 4\alpha'(|Eu_*|^2)(Eu_* : Eu)(Eu_*)_{j,l}\nu_l\Big)_{j=1,\dots,n}\\
&= 2\alpha(|Eu_*|^2)Eu\nu + 4\alpha'(|Eu_*|^2)(Eu_* : Eu)Eu_*\nu, \quad u \in H_p^2(\Omega).
\end{aligned}$$

Hence, the Neumann boundary operator can be written in the form

$$\begin{aligned}
\mathcal{B}_\mathcal{N}(Eu_*)(u,\pi) &= 2\alpha(|Eu_*|^2)Eu\nu + 4\alpha'(|Eu_*|^2)(Eu_* : Eu)Eu_*\nu - \pi\nu,\\
&\qquad u \in H_p^2(\Omega), \quad \pi \in W^{1-\frac{1}{p}}(\Gamma_N),
\end{aligned} \tag{1.5}$$

and the prefect slip boundary operator reads

$$\mathcal{B}_\mathcal{S}(Eu_*)u = 2\alpha(|Eu_*|^2)[Eu\nu]_{\tan} + 4\alpha'(|Eu_*|^2)(Eu_* : Eu)[Eu_*\nu]_{\tan}, \quad u \in H_p^2(\Omega). \tag{1.6}$$

It is worth pointing out, that in the case that $\alpha = \alpha_0$ is constant, we in particular deduce that

$$\mathcal{B}_\mathcal{N}(Eu_*)(u,\pi) = \mathcal{B}_\mathcal{N}(0)(u,\pi) = 2\alpha_0 Eu\nu - \pi\nu \quad \text{and} \quad \mathcal{B}_\mathcal{S}(Eu_*)u = \mathcal{B}_\mathcal{S}(0)u = 2\alpha_0[Eu\nu]_{\tan}. \tag{1.7}$$

We consider the generalized Stokes system

$$\left\{\begin{array}{rcll}
\partial_t u + \mathcal{A}(Eu_*)u + \nabla\pi &=& f & \text{in } (0,T_0)\times\Omega,\\
\operatorname{div} u &=& f_d & \text{in } (0,T_0)\times\Omega,\\
u &=& h_D & \text{on } (0,T_0)\times\Gamma_D,\\
(u\cdot\nu, \mathcal{B}_\mathcal{S}(Eu_*)u) &=& (h_{S,1}, [h_{S,2}]_{\tan}) & \text{on } (0,T_0)\times\Gamma_S,\\
\mathcal{B}_\mathcal{N}(Eu_*)(u,\pi) &=& h_N & \text{on } (0,T_0)\times\Gamma_N,\\
u(0) &=& u_0 & \text{in } \Omega.
\end{array}\right. \tag{1.8}$$

To motivate a natural compatibility condition of the right-hand sides of (1.8), we test the divergence free condition with $\varphi \in {}_0H^1_{p',\Gamma_N}(\Omega)$, where $1 < p' < \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$, to the result

$$\int_\Omega (\operatorname{div} u)\varphi = -\int_\Omega u\cdot\nabla\varphi + \int_{\partial\Omega} u\cdot\nu\varphi = -\int_\Omega u\cdot\nabla\varphi + \int_{\Gamma_D\cup\Gamma_S} u\cdot\nu\varphi.$$

Using $\operatorname{div} u = f_d$ in Ω, $u = h_D$ on Γ_D, and $u \cdot \nu = h_{S,1}$ on Γ_S, it follows that

$$-\int_\Omega u \cdot \nabla\varphi = \int_\Omega f_d\varphi - \int_{\Gamma_D\cup\Gamma_S} (h_D \cdot \nu\chi_{\Gamma_D} + h_{S,1}\chi_{\Gamma_S})\varphi. \tag{1.9}$$

Defining the functional

$$F_{f_d,h}\varphi := \int_\Omega f_d\varphi - \int_{\Gamma_D\cup\Gamma_S} h\varphi, \quad \varphi \in {}_0\widehat{H}^1_{p',\Gamma_N}(\Omega), \tag{1.10}$$

equation (1.9) reads

$$F_{f_d,h_\nu}\varphi = -\int_\Omega u \cdot \nabla\varphi,$$

where $h := h_D \cdot \nu\chi_{\Gamma_D} + h_{S,1}\chi_{\Gamma_S}$. Differentiating with respect to t, we obtain

$$\frac{d}{dt}(F_{f_d,h_\nu}\varphi) = -\int_\Omega (\partial_t u) \cdot \nabla\varphi.$$

If $u \in H^1_p(0,T;L_p(\Omega))$, we conclude that $F_{f_d,h_\nu} \in H^1_p(0,T;{}^0\widehat{H}^{-1}_{p,\Gamma_N}(\Omega))$. We write, following the convention of Bothe and Prüß [BP07, Section 4], the abbreviation $(f_d,h_\nu) \in H^1_p(0,T;{}^0\widehat{H}^{-1}_{p,\Gamma_N}(\Omega))$ for $F_{f_d,h_\nu} \in H^1_p(0,T;{}^0\widehat{H}^{-1}_{p,\Gamma_N}(\Omega))$ and $f_d \in H^1_p(0,T;{}^0\widehat{H}^{-1}_{p,\Gamma_N}(\Omega))$ for $(f_d,0) \in H^1_p(0,T;{}^0\widehat{H}^{-1}_{p,\Gamma_N}(\Omega))$.

Proposition 1.8. *Fix $n \in \mathbb{N}$, $n \geq 2$, $0 < T < T_0$, and $n+2 < p < \infty$. Let $\alpha \in C^{1,1}([0,\infty))$ and $\Omega \subset \mathbb{R}^n$ be a domain with a compact boundary $\partial\Omega$ of class $C^{2,1}$, the half space $\Omega = \mathbb{R}^n_+$, or the whole space $\mathbb{R}^n$. Assume that the boundary $\partial\Omega = \Gamma_D\cup\Gamma_S\cup\Gamma_N$ decomposes in disjoint subsets Γ_D, Γ_S, and Γ_N, where $\Gamma_D, \Gamma_S, \Gamma_N$ are open and closed in $\partial\Omega$. Fix $u_* \in H^1_p(0,T_0;L_p(\Omega)) \cap L_p(0,T_0;H^2_p(\Omega))$. For each*

- $u_0 \in W^{2-\frac{2}{p}}_p(\Omega)$,
- $f \in L_p(0,T;L_p(\Omega))$,
- $f_d \in L_p(0,T;H^1_p(\Omega))$,
- $(f_d,h) \in H^1_p(0,T;{}^0\widehat{H}^{-1}_{p,\Gamma_N}(\Omega))$, *where* $h = h_D \cdot \nu\chi_{\Gamma_D} + h_{S,1}\chi_{\Gamma_S}$ *with* $\operatorname{div} u_0 = f_d(0)$ *in* Ω,
- $h_D \in W^{1-\frac{1}{2p}}_p(0,T;L_p(\Gamma_D)) \cap L_p(0,T;W^{2-\frac{1}{p}}_p(\Gamma_D))$ *with* $u_0 = h_D(0)$ *on* Γ_D,
- $h_{S,1} \in W^{1-\frac{1}{2p}}_p(0,T;L_p(\Gamma_S)) \cap L_p(0,T;W^{2-\frac{1}{p}}_p(\Gamma_S))$ *with* $u_0 \cdot \nu = h_{S,1}(0)$ *on* Γ_S *and* $h_{S,2} \in W^{\frac{1}{2}-\frac{1}{2p}}_p(0,T;L_p(\Gamma_S)) \cap L_p(0,T;W^{1-\frac{1}{p}}_p(\Gamma_S))$ *with* $\mathcal{B}_\mathcal{S}(Eu_*(0))u_0 = h_{S,2}$ *on* Γ_S,
- $h_N \in W^{\frac{1}{2}-\frac{1}{2p}}_p(0,T;L_p(\Gamma_N)) \cap L_p(0,T;W^{1-\frac{1}{p}}_p(\Gamma_N))$ *with* $\mathcal{B}_\mathcal{S}(Eu_*(0))u_0 = h_{N,tan}$ *on* Γ_N,

there exists a unique strong solution (u,π) of (1.8) on $(0,T)$ in the regularity class

- $u \in H^1_p(0,T;L_p(\Omega)) \cap L_p(0,T;H^2_p(\Omega))$,

- $\pi \in L_p(0,T;\widehat{H}^1_p(\Omega))$ *with* $\gamma_{\Gamma_N}\pi \in W_p^{\frac12-\frac{1}{2p}}(0,T;L_p(\Gamma_N)) \cap L_p(0,T;W_p^{1-\frac1p}(\Gamma_N))$.

Further, the solution depends continuously on the data, in the sense that there exists a constant C_T, which depends on T and is independent of the data, such that

$$\begin{aligned}
&\|u\|_{H^1_p(0,T;L_p(\Omega))\cap L_p(0,T;H^2_p(\Omega))} + \|\pi\|_{L_p(0,T;\widehat{H}^1_p(\Omega))} + \|\pi\|_{W_p^{\frac12-\frac{1}{2p}}(0,T;L_p(\Gamma_N))\cap L_p(0,T;W_p^{1-\frac1p}(\Gamma_N))} \\
&\quad \le C_T\Big(\|u_0\|_{W_p^{2-\frac2p}(\Omega)} + \|f\|_{T,\Omega,p,p} + \|f_d\|_{L_p(0,T;H^1_p(\Omega))} + \|(f_d,h)\|_{H^1_p(0,T;\widehat{H}^{-1}_{p,\Gamma_N}(\Omega))} \\
&\quad + \|h_D\|_{W_p^{1-\frac{1}{2p}}(0,T;L_p(\Gamma_D))\cap L_p(0,T;W_p^{2-\frac1p}(\Gamma_D))} + \|h_{S,1}\|_{W_p^{1-\frac{1}{2p}}(0,T;L_p(\Gamma_S))\cap L_p(0,T;W_p^{2-\frac1p}(\Gamma_S))} \\
&\quad + \|h_{S,2}\|_{W_p^{\frac12-\frac{1}{2p}}(0,T;L_p(\Gamma_S))\cap L_p(0,T;W_p^{1-\frac1p}(\Gamma_S))} + \|h_N\|_{W_p^{\frac12-\frac{1}{2p}}(0,T;L_p(\Gamma_S))\cap L_p(0,T;W_p^{1-\frac1p}(\Gamma_S))}\Big).
\end{aligned}$$

In the case $u_0 = 0$ the constant C_T can be chosen independent of T, $0 < T < T_0$.

This was proved by Bothe and Prüß [BP07, Theorem 4.1 and Section 9], considering even more general operators $\mathcal{A}$.

1.2.4 Solvability of the linearisation of a two phase Navier-Stokes equation with surface tension and gravity

This subsection is relevant for Chapter 3, where we analyse a generalized Newtonian two-phase flow problem with surface tension and gravity. To solve the nonlinear problem, we use the Hanzawa transformation to transform the problem onto a fixed domain. We introduce Prüß and Simonett's [PS11, Theorem 3.1] solvability result of the associated linearisation of the transformed problem. This result is used in Chapter 3 to solve the nonlinear problem.

Let $n \in \mathbb{N}$, $n \ge 2$, $\alpha_\pm, \rho_\pm, \sigma, \gamma_a > 0$, and set

$$(\alpha,\rho) = (\alpha_+,\rho_+)\chi_{\mathbb{R}^n_+} + (\alpha_-,\rho_-)\chi_{\mathbb{R}^n_-}.$$

We recall the definition $\dot{\mathbb{R}}^n = \{x = (x', x_n) \in \mathbb{R}^n : x_n \ne 0\}$. The jump of a quantity f is denoted by $[\![f]\!] = \gamma_+ f - \gamma_- f$, where $\gamma_\pm$ denotes the upper and lower trace $\gamma_{\mathbb{R}^n_\pm} f$ respectively. The system under consideration reads

$$(1.11)\qquad \left\{\begin{array}{rcll}
\rho\partial_t u - \alpha\Delta u + \nabla\pi &=& f & \text{in } (0,T_0)\times\dot{\mathbb{R}}^n,\\
\operatorname{div} u &=& f_d & \text{in } (0,T_0)\times\dot{\mathbb{R}}^n,\\
[\![u]\!] &=& 0 & \text{on } (0,T_0)\times\mathbb{R}^{n-1},\\
-[\![\alpha\partial_n u']\!] - [\![\alpha\nabla' u_n]\!] &=& h'_u & \text{on } (0,T_0)\times\mathbb{R}^{n-1},\\
-2[\![\alpha\partial_n u_n]\!] + [\![\pi]\!] - [\![\rho]\!]\gamma_a h - \sigma\Delta' h &=& h_{u,n} & \text{on } (0,T_0)\times\mathbb{R}^{n-1},\\
\partial_t h - u_n &=& h_h & \text{on } (0,T_0)\times\mathbb{R}^{n-1},\\
u(0) &=& u_0 & \text{in } \dot{\mathbb{R}}^n,\\
h(0) &=& h_0 & \text{on } \mathbb{R}^{n-1}.
\end{array}\right.$$

The solvability of (1.11) is established by the next proposition.

Proposition 1.9. *Fix $n \in \mathbb{N}$, $n \ge 2$, $T_0, \rho_\pm, \alpha_\pm, \gamma_a$, and $\sigma > 0$. For each*

- $u_0 \in W_p^{2-\frac2p}(\dot{\mathbb{R}}^n)$ *with* $[\![u_0]\!] = 0$ *on* $\mathbb{R}^{n-1}$,

- $h_0 \in W_p^{3-\frac{2}{p}}(\mathbb{R}^{n-1})$,
- $f \in L_p(0,T_0;L_p(\mathbb{R}^n))$,
- $f_d \in H_p^1(0,T_0;\widehat{H}_p^{-1}(\mathbb{R}^n)) \cap L_p(0,T_0;H_p^1(\dot{\mathbb{R}}^n))$ *with* $f_d(0) = \operatorname{div} u_0$,
- $h_u = (h_u', h_{u,n}) \in W_p^{\frac{1}{2}-\frac{1}{2p}}(0,T_0;L_p(\mathbb{R}^{n-1})) \cap L_p(0,T_0;W_p^{1-\frac{1}{p}}(\mathbb{R}^{n-1}))$ *with* $h_u'(0) = -[\![\alpha\partial_n u_0']\!] - [\![\alpha\nabla' u_{0,n}]\!]$,
- $h_h \in W_p^{1-\frac{1}{2p}}(0,T_0;L_p(\mathbb{R}^{n-1})) \cap L_p(0,T_0;W_p^{2-\frac{1}{p}}(\mathbb{R}^{n-1}))$,

there exists a unique strong solution (u,π,h) *of* (1.11) *on* $(0,T_0)$ *in the regularity class*

- $u \in H_p^1(0,T_0;L_p(\mathbb{R}^n)) \cap L_p(0,T_0;H_p^2(\dot{\mathbb{R}}^n) \cap H_p^1(\mathbb{R}^n))$,
- $\pi \in L_p(0,T_0;\widehat{H}_p^1(\dot{\mathbb{R}}^n))$,
- $[\![\pi]\!] \in W_p^{\frac{1}{2}-\frac{1}{2p}}(0,T_0;L_p(\mathbb{R}^{n-1})) \cap L_p(0,T_0;W_p^{1-\frac{1}{p}}(\mathbb{R}^{n-1}))$,
- $h \in W_p^{2-\frac{1}{2p}}(0,T_0;L_p(\mathbb{R}^{n-1})) \cap H_p^1(0,T_0;W_p^{2-\frac{1}{p}}(\mathbb{R}^{n-1})) \cap L_p(0,T_0;W_p^{3-\frac{1}{p}}(\mathbb{R}^{n-1}))$.

The solution depends continuously on the data, in the sense that there exists a constant C_{T_0}*, which is independent of the data, with*

$$
\begin{aligned}
&\|u\|_{H_p^1(0,T_0;L_p(\mathbb{R}^n))\cap L_p(0,T_0;H_p^2(\dot{\mathbb{R}}^n))} + \|\pi\|_{L_p(0,T_0;\widehat{H}_p^1(\dot{\mathbb{R}}^n))} \\
&\quad + \|[\![\pi]\!]\|_{W_p^{\frac{1}{2}-\frac{1}{2p}}(0,T_0;L_p(\mathbb{R}^{n-1}))\cap L_p(0,T_0;W_p^{1-\frac{1}{p}}(\mathbb{R}^{n-1}))} \\
&\quad + \|h\|_{W_p^{2-\frac{1}{2p}}(0,T_0;L_p(\mathbb{R}^{n-1}))\cap H_p^1(0,T_0;W_p^{2-\frac{1}{p}}(\mathbb{R}^{n-1}))\cap L_p(0,T_0;W_p^{3-\frac{1}{p}}(\mathbb{R}^{n-1}))} \\
&\qquad \leq C_{T_0}\Big(\|u_0\|_{W_p^{2-\frac{2}{p}}(\dot{\mathbb{R}}^n)} + \|h_0\|_{W_p^{3-\frac{2}{p}}(\mathbb{R}^{n-1})} + \|f\|_{T_0,\mathbb{R}^n,p,p} + \|f_d\|_{H_p^1(0,T_0;\widehat{H}_p^{-1}(\mathbb{R}^n))\cap L_p(0,T_0;H_p^1(\dot{\mathbb{R}}^n))} \\
&\qquad\quad + \|h_u\|_{W_p^{\frac{1}{2}-\frac{1}{2p}}(0,T_0;L_p(\mathbb{R}^{n-1}))\cap L_p(0,T_0;W_p^{1-\frac{1}{p}}(\mathbb{R}^{n-1}))} \\
&\qquad\quad + \|h_h\|_{W_p^{1-\frac{1}{2p}}(0,T_0;L_p(\mathbb{R}^{n-1}))\cap L_p(0,T_0;W_p^{2-\frac{1}{p}}(\mathbb{R}^{n-1}))}\Big).
\end{aligned}
$$

For a proof, we refer the reader to Prüß and Simonett [PS11, Theorem 3.1].

1.2.5 Transport equation

The last linear equation analysed in the preliminaries is a transport equation. In viscoelastic fluids, a transport equation describes the evolution of the elastic part of the stress. The solvability result of the transport equation plays an important role in Chapter 2. Let $n \in \mathbb{N}$, $n \geq 2$, and let $\Omega \subset \mathbb{R}^n$ be a domain with a uniform C^2-boundary. The outer normal is denoted by ν. For a given velocity field u, a given right-hand side g, and a given initial value τ_0, we consider

$$
(1.12) \qquad \left\{ \begin{array}{rcll} \partial_t\tau + u.\nabla\tau & = & g & \text{in } (0,T)\times\Omega, \\ \tau(0) & = & \tau_0 & \text{in } \Omega. \end{array} \right.
$$

The aim is to prove the existence of a unique solution τ of (1.12) as well as a-priori estimates.

The transport equation (1.12) is studied intensively in the literature. For a deeper discussion, we refer the reader to Beirão da Veiga [BdV88], DiPerna and Lions [DL89], as well as Novotný [Nov96].

We give a proof of the following proposition. The result and the proof are standard, but to the author's best knowledge this result cannot be found in the literature.

Proposition 1.10. *Fix $n \in \mathbb{N}$, $n \geq 2$, $1 < p < \infty$, $n < q < \infty$, $1 < r \leq \infty$, and $0 < T < T_0$. Let $\Omega \subset \mathbb{R}^n$ be a domain with a uniform C^2-boundary.*

(a) *For each*

- $\tau_0 \in H^1_q(\Omega)$,
- $u \in L_p(0,T;H^2_q(\Omega)) \cap L_r(0,T;L_\infty(\Omega))$ *with* $u \cdot \nu = 0$ *on* $\partial\Omega$,
- $g \in L_1(0,T;H^1_q(\Omega)) \cap L_r(0,T;L_q(\Omega))$,

there exists a unique strong solution τ of (1.12) in the regularity class

- $\tau \in L_\infty(0,T;H^1_q(\Omega)) \cap \widehat{W}^1_r(0,T;L_q(\Omega))$.

Moreover, there exists a constant $C^{(1)}_{Tra}$, which is independent of the data and T, $0 < T < T_0$, such that the estimates

$$\begin{aligned}
\|\tau\|_{L_\infty(0,T;H^1_q(\Omega))} &\leq C^{(1)}_{Tra}\big(\|\tau_0\|_{H^1_q(\Omega)} + \|g\|_{L_1(0,T;H^1_p(\Omega))}\big)e^{C^{(1)}_{Tra}T^{\frac{p-1}{p}}\|u\|_{L_p(0,T;H^2_q(\Omega))}},\\
\|\partial_t\tau\|_{T,\Omega,r,q} &\leq \|g\|_{T,\Omega,r,q} + \|u\|_{T,\Omega,r,\infty}\|\tau\|_{L_\infty(0,T;H^1_q(\Omega))}
\end{aligned}$$

hold.

(b) *Let $\tau \in L_\infty(0,T;H^1_q(\Omega)) \cap \widehat{W}^1_r(0,T;L_q(\Omega))$ be a solution of (1.12), which corresponds to the data*

- $\tau_0 \in H^1_q(\Omega)$,
- $u \in L_p(0,T;H^2_q(\Omega))$ *with* $u \cdot \nu = 0$ *on* $\partial\Omega$,
- $g \in L_1(0,T;L_q(\Omega))$.

Then, there exists a constant $C^{(2)}_{Tra}$, which is independent of the data and T, $0 < T < T_0$, such that the estimate

$$\|\tau\|_{T,\Omega,\infty,q} \leq \big(\|\tau_0\|_{\Omega,q} + \|g\|_{T,\Omega,1,q}\big)e^{C^{(2)}_{Tra}T^{\frac{p-1}{p}}\|\operatorname{div} u\|_{L_p(0,T;H^1_q(\Omega))}}.$$

holds.

We give a proof in Appendix A.

1.3 Fixed point arguments

In this work, we apply three different fixed point arguments to solve nonlinear problems. These are summarized in this section. The first proposition is Schauder's fixed point theorem, the second is the contraction mapping principle, and the third is a modification of the contraction mapping principle, which turns out to be useful in the analysis of viscoelastic fluid models.

Proposition 1.11 (Schauder's fixed point theorem). *Let X be a Banach spaces and $K \subset X$ be a non-empty, compact, and convex subset of X. Every continuous map $\Phi\colon K \to K$ admits at least one fixed point.*

If (K, d) is a metric space, we call a map $\Phi\colon K \to K$ a contraction mapping, if there exists a constant $\delta < 1$, such that

$$d(\Phi(x), \Phi(y)) < \delta d(x, y), \quad x, y \in K$$

holds.

Proposition 1.12 (Contraction mapping principle). *Let (K, d) be a non-empty, complete metric space. Every contraction mapping $\Phi\colon K \to K$ admits one and only one fixed point.*

Proposition 1.13. *Fix two Banach spaces X, X^w with $X \hookrightarrow X^w$ and $\delta < 1$. Let either X be reflexive or let X admit a separable pre-dual and let $K \subset X$ be non-empty, convex, closed, and bounded subset of X. Every map $\Phi\colon X \to X$ with $\Phi(K) \subset K$ and*

$$\|\Phi(x) - \Phi(y)\|_{X^w} \leq \delta \|x - y\|_{X^w}, \quad x, y \in K$$

admits one and only one fixed point in K.

For a proof of this variant of the contraction mapping principle, we refer the reader to Kremel and Pokorný [KP09, Lemma 2.5].

1.4 Function spaces, embedding and trace theorems, and Nemytskij operators

In this section, we introduce some function spaces, embedding properties of these spaces and we consider Nemytskij operators on these spaces. These results are used frequently in the following chapters.

Fix $T > 0$, $n \in \mathbb{N}$, and $n \geq 2$. Let $\Omega \subset \mathbb{R}^n$ be a domain with a uniform C^2-boundary and let $\Gamma \subset \partial\Omega$ an open and closed subset of the boundary $\partial\Omega$. We define the solution space for the velocity field

$$\mathbb{E}_u^{p,q}(T, \Omega) \;\; := \;\; H_p^1(0, T; L_q(\Omega)) \cap L_p(0, T; H_q^2(\Omega)), \quad 1 < p, q < \infty.$$

The gradient of function $u \in \mathbb{E}_u^{p,q}(T, \Omega)$ belongs to the space

$$\mathbb{E}_u^{p,q,w}(T, \Omega) \;\; := \;\; H_p^{\frac{1}{2}}(0, T; L_q(\Omega)) \cap L_p(0, T; H_q^1(\Omega)), \quad 1 < p, q < \infty.$$

In the case of inhomogeneous boundary conditions, we consider the case $p = q$. We set

$$\begin{aligned}
\mathbb{E}_u(T,\Omega) &:= \mathbb{E}_u^{p,p}(T,\Omega), \qquad 1<p<\infty,\\
\mathbb{E}_u^w(T,\Omega) &:= \mathbb{E}_u^{p,p,w}(T,\Omega), \qquad 1<p<\infty.
\end{aligned}$$

Moreover, we define the solution space for the height function h, which appears in (1.11) (see Theorem 1.9), i.e.

$$\mathbb{E}_h(T,\mathbb{R}^{n-1}) := W_p^{2-\frac{1}{2p}}(0,T;L_p(\mathbb{R}^{n-1})) \cap H_p^1(0,T;W_p^{2-\frac{1}{p}}(\mathbb{R}^{n-1})) \cap L_p(0,T;W_p^{3-\frac{1}{p}}(\mathbb{R}^{n-1})).$$

Investigating inhomogeneous boundary conditions, we consider the Dirichlet trace $\gamma_\Gamma u$ and the Neumann trace $\gamma_\Gamma \nabla u$ of a function $u \in \mathbb{E}_u(T,\Omega)$. For this purpose, we define

$$\begin{aligned}
\mathbb{H}_h(T,\Gamma) &:= W_p^{1-\frac{1}{2p}}(0,T;L_p(\Gamma)) \cap L_p(0,T;W_p^{2-\frac{1}{p}}(\Gamma)), \quad 1<p<\infty,\\
\mathbb{H}_u(T,\Gamma) &:= W_p^{\frac{1}{2}-\frac{1}{2p}}(0,T;L_p(\Gamma)) \cap L_p(0,T;W_p^{1-\frac{1}{p}}(\Gamma)), \quad 1<p<\infty.
\end{aligned}$$

If the boundary of the domain Ω is compact or $\Omega = \mathbb{R}^n_+$ is a half space, it is know that $\gamma_\Gamma u \in \mathbb{H}_h(T,\Gamma)$ and $\gamma_\Gamma \nabla u \in \mathbb{H}_u(T,\Gamma)$ (see Proposition 1.15).

Furthermore, considering a shear-rate dependent viscosity, terms of the form $\alpha(|\gamma_\Gamma Eu|^2)$ appear on the boundary. In general $\alpha(|\gamma_\Gamma Eu|^2) \notin \mathbb{H}_u(T,\Gamma)$ for $u \in \mathbb{E}_u(T,\Omega)$. To treat this term, we introduce the auxiliary space

$$\mathbb{H}_u^\infty(T,\Gamma) := \{u \in BUC([0,T],BUC(\Gamma))\colon \|u\|_{\mathbb{H}_u^\infty(T,\Gamma)} = \|u\|_{T,\Gamma,\infty,\infty} + [u]_{\mathbb{H}_u(T,\Gamma)} < \infty\}, \qquad 1<p<\infty,$$

with

$$\begin{aligned}
[u]_{\mathbb{H}_u(T,\Gamma)} &:= \Big(\int_0^T\int_0^T \frac{\|u(t)-u(s)\|_{p,\Gamma}^p}{|t-s|^{\frac{1}{2}+\frac{p}{2}}}dsdt\Big)^{\frac{1}{p}} + \Big(\int_0^T\int_\Gamma\int_\Gamma \frac{|u(t,x)-u(t,y)|^p}{|x-y|^{n-2+p}}dxdydt\Big)^{\frac{1}{p}}\\
&= \Big(\int_\Gamma [u(\cdot,x)]^p_{W_p^{\frac{1}{2}-\frac{1}{2p}}(0,T)}dx\Big)^{\frac{1}{p}} + \Big(\int_0^T [u(t,\cdot)]^p_{W_p^{1-\frac{1}{p}}(\Gamma)}dt\Big)^{\frac{1}{p}},
\end{aligned} \tag{1.13}$$

where we applied Tonelli's theorem. If the boundary of the domain Ω is compact or $\Omega = \mathbb{R}^n_+$ is a half space, we are able to show that $\alpha(|\gamma_\Gamma Eu|^2) \in \mathbb{H}_u^\infty(T,\Gamma)$ for $u \in \mathbb{E}_u(T,\Omega)$, provided $p > n+2$ (see Proposition 1.17). Moreover we have the embedding $\mathbb{H}_u(T,\Gamma) \hookrightarrow \mathbb{H}_u^\infty(T,\Gamma)$, if $p > n+2$ (see Proposition 1.14).

Furthermore, we define the spaces with vanishing initial value, i.e. for

$$X \in \{\mathbb{E}_u^{p,q}(T,\Omega), \mathbb{E}_u^{p,q,w}(T,\Omega), \mathbb{E}_u(T,\Omega), \mathbb{E}_u^w(T,\Omega), \mathbb{E}_h(T,\Gamma), \mathbb{H}_h(T,\Gamma), \mathbb{H}_u(T,\Gamma)\},$$

we set, if the time trace exists,

$${}_0X = \{x \in X\colon x(0)\}.$$

The next proposition summarizes embedding theorems.

Proposition 1.14 (on embedding theorems)**.** *Fix $n \in \mathbb{N}$, $n \geq 2$, $1 < p, q < \infty$, and $0 < T < T_0$. Let $\Omega \subset \mathbb{R}^n$ be a domain with a uniform C^2-boundary and let $\Gamma \subset \partial\Omega$ be an open and compact subset of the boundary. Then*

$$\mathbb{E}_u^{p,q}(T,\Omega) \hookrightarrow L_{3p}(0,T;L_{3q}(\Omega)) \cap L_{\frac{3p}{2}}(0,T;H^1_{\frac{3q}{2}}(\Omega)), \qquad 1 < p < \infty, \quad n < q < \infty,$$

and

$$\begin{aligned} \mathbb{E}_u^{p,q}(T,\Omega) &\hookrightarrow BUC([0,T],BUC^1(\overline{\Omega})) \cap BUC([0,T],H^1_q(\Omega)), && \tfrac{1}{p} + \tfrac{n}{2q} < \tfrac{1}{2}, \\ \mathbb{E}_u^{p,q,w}(T,\Omega) &\hookrightarrow BUC([0,T],BUC(\overline{\Omega})) \cap BUC([0,T],L_q(\Omega)), && \tfrac{1}{p} + \tfrac{n}{2q} < \tfrac{1}{2}, \end{aligned}$$

as well as

$$\mathbb{H}_u(T,\Gamma) \hookrightarrow BUC([0,T],BUC(\Gamma)), \qquad n+2 < p < \infty.$$

In addition, the embedding constants can be chosen uniformly in T, $0 < T < T_0$, provided that the functions vanish at $t = 0$. Further, the embedding

$$\mathbb{E}_u^{p,q,w}(T,\Omega) \hookrightarrow L_{\frac{3p}{2}}(0,T;L_{\frac{3q}{2}}(\Omega)), \qquad 1 < p < \infty, \quad n < q < \infty,$$

holds, and the embedding constant can be chosen uniformly in T, $0 < T < T_0$, provided that $p \neq 2$ and, if the time trace exists, the functions vanish at $t = 0$. Moreover,

$$\mathbb{E}_h(T_0,\mathbb{R}^{n-1}) \hookrightarrow BUC^1([0,T_0],BUC^1(\mathbb{R}^{n-1})) \cap BUC([0,T_0],BUC^2(\mathbb{R}^{n-1})), \quad n+2 < p < \infty.$$

Proof. For a proof of the first embedding, i.e.

$$\mathbb{E}_u^{p,q}(T,\Omega) \hookrightarrow L_{3p}(0,T;L_{3q}(\Omega)) \cap L_{\frac{3p}{2}}(0,T;H^1_{\frac{3q}{2}}(\Omega)), \qquad 1 < p < \infty, \quad n < q < \infty,$$

as well as the estimate

$$\|u\|_{L_{3p}(0,T;L_{3q}(\Omega)) \cap L_{\frac{3p}{2}}(0,T;H^1_{\frac{3q}{2}}(\Omega))} \leq C\|u\|_{{}_0\mathbb{E}_u^{p,q}(T,\Omega)}, \quad u \in {}_0\mathbb{E}_u^{p,q}(T,\Omega), \quad 0 < T < T_0,$$

we refer to Dintelmann, Geissert, and Hieber [DGH09, Lemma 4.2]. Using similar arguments, we prove the embeddings

(1.14)

$$\begin{aligned} \mathbb{E}_u^{p,q,w}(T,\Omega) &\hookrightarrow L_{\frac{3p}{2}}(0,T;L_{\frac{3q}{2}}(\Omega)), && 1 < p < \infty, \quad n < q < \infty, \\ \mathbb{E}_u^{p,q}(T,\Omega) &\hookrightarrow BUC([0,T],BUC^1(\overline{\Omega})) \cap BUC([0,T],H^1_q(\Omega)), && \tfrac{1}{p} + \tfrac{n}{2q} < \tfrac{1}{2}, \\ \mathbb{E}_u^{p,q,w}(T,\Omega) &\hookrightarrow BUC([0,T],BUC(\overline{\Omega})) \cap BUC([0,T],L_q(\Omega)), && \tfrac{1}{p} + \tfrac{n}{2q} < \tfrac{1}{2}: \end{aligned}$$

Due to Adams and Fournier [AF03, Theorem 5.24], there exists a total extension operator for Ω. Therefore, we obtain

$$\begin{aligned} \mathbb{E}_u^{p,q,w}(T,\Omega) &\hookrightarrow H_p^{\frac{\alpha}{2}}(0,T;H_q^{1-\alpha}(\Omega)), && \alpha \in (0,1), \\ \mathbb{E}_u^{p,q}(T,\Omega) &\hookrightarrow H_p^{\alpha}(0,T;H_q^{2-2\alpha}(\Omega)), && \alpha \in (0,1), \end{aligned}$$

by the mixed derivative theorem (see Sobolevskii [Sob75]) and hence, we deduce that

$$\begin{array}{rcll} H_p^{\frac{\alpha}{2}}(0,T;H_q^{1-\alpha}(\Omega)) & \hookrightarrow & L_{\frac{3p}{2}}(0,T;L_{\frac{3q}{2}}(\Omega)), & \frac{2}{3p}<\alpha<1-\frac{n}{3q}, \\ H_p^{\frac{\alpha}{2}}(0,T;H_q^{1-\alpha}(\Omega)) & \hookrightarrow & BUC([0,T];BUC(\overline{\Omega})), & \frac{2}{p}<\alpha<1-\frac{n}{q}, \\ H_p^{\alpha}(0,T;H_q^{2-2\alpha}(\Omega)) & \hookrightarrow & BUC([0,T];BUC^1(\overline{\Omega})), & \frac{1}{p}<\alpha<\frac{1}{2}-\frac{n}{2q}, \end{array}$$

by Sobolev's embedding theorem. This shows the embeddings (1.14). Next, we show that the embedding constant can be chosen independent of T, $0<T<T_0$, provided the initial values vanish. This is shown exemplary for the embedding ${}_0\mathbb{E}_u^{p,q}(T,\Omega)\hookrightarrow BUC([0,T];BUC^1(\overline{\Omega}))$. Let $1<p,q<\infty$ with $\frac{1}{p}+\frac{n}{2q}<\frac{1}{2}$, $0<T<T_0$, and $u\in{}_0\mathbb{E}_u^{p,q}(T,\Omega)$. We define

$$\bar{u}(t)=\begin{cases}0 & t\in(0,T_0-T]\\ u(t+T-T_0) & t\in(T_0-T,T_0)\end{cases},\quad t\in(0,T_0).$$

By construction, it follows that $\bar{u}\in\mathbb{E}_u(T_0,\Omega)$ and hence, we have $\bar{u}\in BUC([0,T_0];BUC^1(\overline{\Omega}))$. Furthermore, we deduce that

$$\|u\|_{BUC([0,T];BUC^1(\overline{\Omega}))}=\|\bar{u}\|_{BUC([0,T_0];BUC^1(\overline{\Omega}))}\le C\|\bar{u}\|_{\mathbb{E}_u^{p,q}(T_0,\Omega)}=C\|u\|_{\mathbb{E}_u^{p,q}(T,\Omega)}.$$

With the same argumentation, we infer for the admissble values of p and q the estimates

$$\begin{array}{rcll} \|u\|_{L_{\frac{3p}{2}}(0,T;L_{\frac{3q}{2}}(\Omega))} & \le & C\|u\|_{{}_0\mathbb{E}_u^{p,q,w}(T,\Omega)}, & u\in{}_0\mathbb{E}_u^{p,q,w}(T,\Omega),\quad 0<T<T_0, \\ \|u\|_{BUC([0,T];BUC(\overline{\Omega}))} & \le & C\|u\|_{{}_0\mathbb{E}_u^{p,q,w}(T,\Omega)}, & u\in{}_0\mathbb{E}_u^{p,q,w}(T,\Omega),\quad 0<T<T_0. \end{array}$$

Prüß and Simonett [PS11, Proposition 5.1 (b) and (d)] proved the embeddings

$$\mathbb{H}_u(T,\mathbb{R}^{n-1})\hookrightarrow BUC([0,T],BUC(\mathbb{R}^{n-1})),\quad n+2<p<\infty, \tag{1.15}$$

and

$$\mathbb{E}_h(T_0,\mathbb{R}^{n-1})\hookrightarrow BUC^1([0,T_0],BUC^1(\mathbb{R}^{n-1}))\cap BUC([0,T_0],BUC^2(\mathbb{R}^{n-1})),\quad n+2<p<\infty,$$

as well as the fact, that the embedding constants in (1.15) can be chosen uniformly in T, $0<T<T_0$, provided the functions vanish at $t=0$. Parameterizing the manifold Γ, we obtain the embedding

$$\mathbb{H}_u(T,\Gamma)\hookrightarrow BUC([0,T],BUC(\Gamma)),\quad n+2<p<\infty,$$

and

$$\|h\|_{BUC([0,T],BUC(\Gamma))}\le C\|h\|_{{}_0\mathbb{H}_u(T,\Gamma)},\quad h\in{}_0\mathbb{H}_u(T,\Gamma),\quad 0<T<T_0,$$

by the same arguments. □

Trace and extension theorems are summarized in the following proposition.

Proposition 1.15 (on trace and extension operators)**.** *Fix $n\in\mathbb{N}$, $n\ge 2$, $0<T<T_0$, and $n+2<p<\infty$. Let $\Omega\subset\mathbb{R}^n$ be a domain with a compact C^2-boundary, or the half space $\Omega=\mathbb{R}^n_+$ and let $\Gamma\subset\partial\Omega$ be an open and closed subset of the boundary $\partial\Omega$.*

(a) *The trace operators*

$$\gamma_\Gamma\colon \mathbb{E}_u(T,\Omega) \to \mathbb{H}_h(T,\Gamma), \quad \gamma_\Gamma\nabla\colon \mathbb{E}_u(T,\Omega) \to \mathbb{H}_u(T,\Gamma), \quad \text{and} \quad \gamma_\Gamma\colon \mathbb{E}_u^w(T,\Omega) \to \mathbb{H}_u(T,\Gamma)$$

are continuous. In addition, the operator norm is uniformly bounded in T, $0 < T < T_0$, provided that the functions vanish at $t = 0$.

(b) *There exist continuous extension operators*

$$\begin{aligned} \mathcal{E}_t\colon W_p^{1-\frac{3}{p}}(\Gamma) &\to \mathbb{H}_u(T_0,\Gamma), \\ \mathcal{E}_t\colon W_p^{2-\frac{3}{p}}(\Gamma) &\to \mathbb{H}_h(T_0,\Gamma). \end{aligned}$$

Proof. If the boundary of the domain Ω is compact, the first two embeddings of assertion **(a)** is a special case of Ladyzhenskaya, Solonnikov, and Ural'tseva [LSU68, Lemma 3.4] as well as Weidemaier [Wei94, Theorem 1], where also mixed L_p-norms are considered and the third embedding is a special case of Meyries and Schnaubelt [MS12, Theorem 4.5], where also Sobolev spaces with weights are investigated. Denk, Hieber, and Prüß [DHP07, Lemma 3.5] proved

$$\gamma_\Gamma\colon \mathbb{E}_u(T,\mathbb{R}^n_+) \to \mathbb{H}_h(T,\mathbb{R}^{n-1}) \quad \text{and} \quad \gamma_\Gamma\colon \mathbb{E}_u^w(T,\mathbb{R}^n_+) \to \mathbb{H}_u(T,\mathbb{R}^{n-1}).$$

Due to the mixed derivative theorem (see [Sol77b]), the map $\nabla\colon \mathbb{E}_u(T,\mathbb{R}^n_+) \to \mathbb{E}_u^w(T,\mathbb{R}^n_+)$ is continuous and hence

$$\gamma_\Gamma\nabla\colon \mathbb{E}_u(T,\mathbb{R}^n_+) \to \mathbb{H}_u(T,\mathbb{R}^{n-1})$$

is continuous.

Next, we show that we can choose the embedding constant of $\gamma_\Gamma\colon {}_0\mathbb{E}_u(T,\Omega) \to {}_0\mathbb{H}_u(T,\Gamma)$ independent of T, $0 < T < T_0$. Let $u \in {}_0\mathbb{E}_u(T,\Omega)$ and we define

$$\bar{u}(t) = \begin{cases} 0 & t \in (0, T_0 - T] \\ u(t + T - T_0) & t \in (T_0 - T, T_0) \end{cases}, \quad t \in (0, T_0).$$

Then, $\bar{u} \in \mathbb{E}_u(T_0,\Omega)$ and we have $\|\gamma_\Gamma\bar{u}\|_{\mathbb{H}_u(T_0,\Gamma)} \le C\|\bar{u}\|_{\mathbb{E}_u(T_0,\Omega)}$. By the construction of $\bar{u}$, this implies

$$\|\gamma_\Gamma u\|_{\mathbb{H}_u(T,\Gamma)} = \|\gamma_\Gamma\bar{u}\|_{\mathbb{H}_u(T_0,\Gamma)} \le C\|\bar{u}\|_{\mathbb{E}_u(T_0,\Omega)} = C\|u\|_{\mathbb{E}_u(T,\Omega)}.$$

By the same arguments, it follows that

$$\begin{aligned} \|\gamma_\Gamma\nabla u\|_{{}_0\mathbb{H}_u(T,\Gamma)} &\le C\|u\|_{{}_0\mathbb{E}_u(T,\Omega)}, \quad u \in {}_0\mathbb{E}_u(T,\Omega), \quad 0 < T < T_0, \\ \|\gamma_\Gamma u\|_{{}_0\mathbb{H}_u(T,\Gamma)} &\le C\|u\|_{{}_0\mathbb{E}_u^w(T,\Omega)}, \quad u \in {}_0\mathbb{E}_u^w(T,\Omega), \quad 0 < T < T_0. \end{aligned}$$

Denk, Saal, and Seiler [DSS08, Theorem 4.5] proved the existence of a continuous extension operator

$$\begin{aligned} \mathcal{E}_t\colon W_p^{1-\frac{3}{p}}(\mathbb{R}^{n-1}) &\to \mathbb{H}_u(T_0,\mathbb{R}^{n-1}), \\ \mathcal{E}_t\colon W_p^{2-\frac{3}{p}}(\mathbb{R}^{n-1}) &\to \mathbb{H}_h(T_0,\mathbb{R}^{n-1}). \end{aligned}$$

Parameterizing the manifold Γ, the result carries over to the general case.

$\square$

Results on pointwise multiplications are summarized in the next proposition.

Proposition 1.16 (on pointwise multiplications)**.** *Fix $n \in \mathbb{N}$, $n \geq 2$, $0 < T < T_0$, and let $n+2 < p < \infty$. Let $\Omega \subset \mathbb{R}^n$ be a uniform C^2-domain and $\Gamma \subset \partial\Omega$ be an open and closed subset of the boundary. Then, $W_p^{1-\frac{3}{p}}(\Gamma)$, $W_p^{1-\frac{1}{p}}(\Gamma)$, $\mathbb{H}_u(T,\Gamma)$, $\mathbb{H}_u^\infty(T,\Gamma)$, and $\mathbb{H}_h(T_0,\mathbb{R}^{n-1})$ are multiplication algebras. Furthermore,*

$$\mathbb{H}_u(T,\Gamma) \cdot \mathbb{H}_u^\infty(T,\Gamma) \quad \hookrightarrow \quad \mathbb{H}_u(T,\Gamma).$$

Moreover, there exists a constant C uniformly in T, $0 < T < T_0$, such that

$$\begin{array}{rcll} \|fg\|_{\mathbb{H}_u^\infty(T,\Gamma)} & \leq & C\|f\|_{\mathbb{H}_u^\infty(T,\Gamma)}\|g\|_{\mathbb{H}_u^\infty(T,\Gamma)}, & f,g \in \mathbb{H}_u^\infty(T,\Gamma), \\ \|fg\|_{{}_0\mathbb{H}_u(T,\Gamma)} & \leq & C\|f\|_{\mathbb{H}_u^\infty(T,\Gamma)}\|g\|_{{}_0\mathbb{H}_u(T,\Gamma)}, & f \in \mathbb{H}_u^\infty(T,\Gamma), \quad g \in {}_0\mathbb{H}_u(T,\Gamma), \\ \|fg\|_{{}_0\mathbb{H}_u(T,\Gamma)} & \leq & C\|f\|_{{}_0\mathbb{H}_u(T,\Gamma)}\|g\|_{{}_0\mathbb{H}_u(T,\Gamma)}, & f,g \in {}_0\mathbb{H}_u(T,\Gamma). \end{array}$$

Proof. Let $0 < T < T_0$ and $n+2 < p < \infty$. Due to Prüß and Simonett [PS10, Lemma 6.1], $\mathbb{H}_u(T,\mathbb{R}^{n-1})$ and $\mathbb{H}_h(T,\mathbb{R}^{n-1})$ are multiplication algebras and due to Runst and Sickel [RS96, 4.6.4 Theorem 1], $W^{1-\frac{3}{p}}(\mathbb{R}^{n-1})$ and $W^{1-\frac{1}{p}}(\mathbb{R}^{n-1})$ are multiplication algebras. We apply the same methods to prove the proposition in the case $\Gamma \neq \mathbb{R}^{n-1}$.

First, we assume that $(G,m) \in \{(\Gamma, n-1), ((0,T),1)\}$, $0 < s < 1$, and $\tilde{f}, \tilde{g} \in \widehat{W}_p^s(G)$. We compute

$$\begin{aligned} [\tilde{f}\cdot\tilde{g}]_{W_p^s(G)} &= \Big(\int_G\int_G \frac{|(\tilde{f}\tilde{g})(x) - (\tilde{f}\tilde{g})(y)|^p}{|x-y|^{m+sp}}dxdy\Big)^{\frac{1}{p}} \\ &= \Big(\int_G\int_G \frac{|(\tilde{f}(x)(\tilde{g}(x) - \tilde{g}(y)) + (\tilde{f}(x) - \tilde{f}(y))\tilde{g}(y)|^p}{|x-y|^{m+sp}}dxdy\Big)^{\frac{1}{p}} \\ &\leq \|\tilde{f}\|_{G,\infty}[\tilde{g}]_{W_p^s(G)} + [\tilde{f}]_{W_p^s(G)}\|\tilde{g}\|_{G,\infty}. \end{aligned} \tag{1.16}$$

By the embeddings $W_p^{1-\frac{3}{p}}(\Gamma) \hookrightarrow L_\infty(\Gamma)$ and $W_p^{1-\frac{1}{p}}(\Gamma) \hookrightarrow L_\infty(\Gamma)$, we infer the algebra property of $W_p^{1-\frac{3}{p}}(\Gamma)$ and $W_p^{1-\frac{1}{p}}(\Gamma)$.

Let from now on $f, g \in \mathbb{H}_u^\infty(T,\Gamma)$. Applying (1.13) and (1.16), it follows that

$$\begin{aligned} [f\cdot g]_{\mathbb{H}_u(T,\Gamma)} &= \Big(\int_\Gamma [(f\cdot g)(\cdot,x)]^p_{W_p^{\frac{1}{2}-\frac{1}{2p}}(0,T)}dx\Big)^{\frac{1}{p}} + \Big(\int_0^T [(f\cdot g)(t,\cdot)]^p_{W_p^{1-\frac{1}{p}}(\Gamma)}dt\Big)^{\frac{1}{p}} \\ &\leq \Big(\int_\Gamma (\|f(\cdot,x)\|_{(0,T),\infty}[g(\cdot,x)]_{W_p^{\frac{1}{2}-\frac{1}{2p}}(0,T)} + [f(\cdot,x)]_{W_p^{\frac{1}{2}-\frac{1}{2p}}(0,T)}\|g(\cdot,x)\|_{(0,T),\infty})^p dx\Big)^{\frac{1}{p}} \\ &\quad + \Big(\int_\Gamma (\|f(t,\cdot)\|_{\Gamma,\infty}[g(t,\cdot)]_{W_p^{1-\frac{1}{p}}(\Gamma)} + [f(t,\cdot)]_{W_p^{1-\frac{1}{p}}(\Gamma)}\|g(t,\cdot)\|_{\Gamma,\infty})^p dt\Big)^{\frac{1}{p}} \\ &\leq C(\|f\|_{T,\Gamma,\infty,\infty}[g]_{\mathbb{H}_u(T,\Gamma)} + [f]_{\mathbb{H}_u(T,\Gamma)}\|g\|_{T,\Gamma,\infty,\infty}). \end{aligned} \tag{1.17}$$

Combining the result of the previous calculation and the proposition on embedding theorems (Proposition 1.14) yields the assertion. ☐

The next subject are the mapping properties of Nemytskij operators in the spaces introduced above. For $N \in \mathbb{N}$ and $\Psi\colon \mathbb{R}^N \to \mathbb{R}$, we define the corresponding Nemytskij operator via

$$\Psi(f) := \Psi \circ f, \quad f\colon \Omega \to \mathbb{R}^N.$$

Proposition 1.17 (on Nemytskij operators)**.** *Fix $n, N \in \mathbb{N}$, $n \geq 2$, $n+2 < p < \infty$, $0 < T < T_0$, and $s \in \{1-\frac{3}{p}, 1-\frac{1}{p}\}$. Let $\Omega \subset \mathbb{R}^n$ be a domain with a compact C^2-boundary, the half space $\Omega = \mathbb{R}^n_+$, or the whole space $\Omega = \mathbb{R}^n$, and let $\Gamma \subset \partial\Omega$ be an open and closed subset of the boundary $\partial\Omega$.*

(a) *Let $R_0 > 0$ and $\Psi \in C^1(\mathbb{R}^N)$. There exists a constant C, such that for all $0 < T < T_0$ the estimates*

$$\begin{array}{lcll} \|\Psi(f)\|_{\mathbb{H}^\infty_u(T,\Gamma)} & \leq & C & f \in B_{\mathbb{H}^\infty_u(T,\Gamma)}(0, R_0), \\ \|\Psi(f) - \Psi(0)\|_{L_\infty(0,T;H^1_p(\Omega))} & \leq & C & f \in B_{L_\infty(0,T;H^1_p(\Omega))}(0, R_0) \end{array}$$

hold.

(b) *The Nemytskij operators*

$$\begin{array}{llll} \Psi\colon W^s_p(\Gamma) & \to & \widehat{W}^s_p(\Gamma) \cap L_\infty(\Gamma), & \Psi \in C^3(\mathbb{R}^N), \\ \Psi\colon \mathbb{H}_u(T,\Gamma) & \to & \mathbb{H}^\infty_u(T,\Gamma), & \Psi \in C^3(\mathbb{R}^N), \\ \Psi\colon BUC([0,T], BUC(\overline{\Omega})) & \to & BUC([0,T], BUC(\overline{\Omega})), & \Psi \in C^2(\mathbb{R}^N), \\ \Psi\colon BUC([0,T], BUC(\dot{\mathbb{R}}^n)) & \to & BUC([0,T], BUC(\dot{\mathbb{R}}^n)) & \Psi \in C^2(\mathbb{R}^N), \\ \Psi\colon L_\infty(0,T; H^1_p(\Omega)) & \to & L_\infty(0,T; H^1_p(\Omega)), & \Psi \in C^3(\mathbb{R}^N), \quad \Psi(0) = 0 \end{array}$$

are continuously Fréchet differentiable. In each case, the Fréchet derivative is the multiplication operator

$$D\Psi(f)h = \Psi'(f)h, \quad f, h \in X,$$

where X stands in each case for the corresponding Banach space

$$W^s_p(\Omega), \mathbb{H}_u(T,\Gamma), BUC([0,T], BUC(\overline{\Omega})), BUC([0,T], BUC(\dot{\mathbb{R}}^n)), \textit{or } L_\infty(0,T; H^1_p(\Omega)).$$

We give a proof in Appendix B.

Remark 1.18. The order of differentiability of the function Ψ in the previous proposition is not optimal, but sufficient for our purpose.

Proposition 1.19. *Fix $n, N \in \mathbb{N}$, $n \geq 2$, $0 < T < T_0$, $R_0 > 0$, $n+2 < p < \infty$. Let $\Omega \subset \mathbb{R}^n$ be a domain with a compact C^2-boundary, the half space $\Omega = \mathbb{R}^n$, or the whole space $\Omega = \mathbb{R}^n$, and let $\Gamma \subset \partial\Omega$ be an open and closed subset of the boundary. Assume $\Psi_1, \Psi_2 \in C^3(\mathbb{R}^N)$ and $\Psi_3 \in C^3(\mathbb{R})$. There exists a constant C, such that for all $0 < T < T_0$ the estimates*

$$\|\Psi_1(f) - \Psi_1(g)\|_{L_\infty(0,T;H^1_p(\Omega))} \leq C\|f-g\|_{L_\infty(0,T;H^1_p(\Omega))}, \quad f, g \in B_{L_\infty(0,T;H^1_p(\Omega))}(0, R_0),$$

and

$$\|\Psi_2(f) - \Psi_2(g)\|_{{}_0\mathbb{H}_u(T,\Gamma)} \leq C\|f-g\|_{{}_0\mathbb{H}_u(T,\Gamma)},$$
$$f, g \in B_{\mathbb{H}_u(T,\Gamma) \cap L_\infty(0,T;L_\infty(\Gamma))}(0, R_0), \quad f - g \in {}_0\mathbb{H}_u(T,\Gamma),$$

as well as

$$\|\Psi_3(f) - \Psi_3(g) - \Psi_3'(g)(f-g)\|_{{}_0\mathbb{H}_u(T,\Gamma)} \le C\|f-g\|^2_{{}_0\mathbb{H}_u(T,\Gamma)},$$
$$f, g \in B_{\mathbb{H}_u(T,\Gamma)\cap L_\infty(0,T;L_\infty(\Gamma))}(0,R_0), \quad f-g \in {}_0\mathbb{H}_u(T,\Gamma)$$

hold.

We give a proof in Appendix B.

1.5 Useful inequalities

In Section 2.1 we use an energy argument to show the uniqueness of a solution. A crucial point is Korn's second inequality and Gronwall's Lemma.

Proposition 1.20 (Korn's second inequality)**.** *Let Ω be a bounded domain with a C^1-boundary. There exists a constant $C > 0$, such that*

$$\|u\|_{H^1_2(\Omega)} \le C(\|\nabla u + (\nabla u)^T\|_{\Omega,2} + \|u\|_{\Omega,2})$$

holds.

For a proof, we refer the reader to Nitsche [Nit81].

Proposition 1.21 (Gronwall's Lemma)**.** *Let $0 < T < \infty$, $a \in L_\infty(0,T)$, $b \in C([0,T])$ monotone increasing, and $c \in L_1(0,T)$ with $c > 0$ almost everywhere. Assume that*

$$a(t) \le b(t) + \int_0^t c(s)a(s)ds, \quad \textit{almost everywhere in } (0,T).$$

Then

$$a(t) \le b(t)e^{\int_0^t c(s)ds}, \quad \textit{almost everywhere in } (0,T).$$

For a proof, we refer the reader to Emmrich [Emm04, Lemma 7.3.1].

Proposition 1.22 (Mean value theorem)**.** *Let X, Y be two Banach spaces and $U \subset X$ be an open subset. Assume that $F\colon U \to Y$ is Fréchet differentiable, $f, h \in X$, and*

$$I := \{f + \lambda h \colon 0 \le \lambda \le 1\} \subset U.$$

Then

$$\|F(f+h) - F(f)\|_X \le \sup_{z\in I} \|DF(z)\|_{\mathcal{L}(X)}\|h\|_X.$$

For a proof, we refer the reader to Werner [Wer05, Theorem III.5.4].

Chapter 2

Generalized viscoelastic fluids on fixed domains

In this chapter, we analyse a mathematical model for an incompressible generalized viscoelastic fluid on a fixed, not necessarily bounded domain $\Omega \subset \mathbb{R}^n$. We assume that the boundary of the domain $\partial\Omega = \Gamma_D \cup \Gamma_S$ decomposes into two disjoint subsets, Γ_D and Γ_S, which are open and closed in $\partial\Omega$. The outer normal is denoted by ν. The aim is to prove local-in-time solvability for arbitrarily large initial data for the following system:

$$
\left\{
\begin{array}{rcll}
\rho(\partial_t u + u.\nabla u) - \operatorname{Div} 2\alpha(|Eu|^2)Eu + \nabla\pi &=& \operatorname{Div}\mu(\tau) + f & \text{in } (0,T_0)\times\Omega, \\
\operatorname{div} u &=& 0 & \text{in } (0,T_0)\times\Omega, \\
\partial_t\tau + u.\nabla\tau &=& g(\nabla u, \tau) & \text{in } (0,T_0)\times\Omega, \\
u &=& 0 & \text{on } (0,T_0)\times\Gamma_D, \\
(u\cdot\nu, [2\alpha(|Eu|^2)Eu\nu + \mu(\tau)\nu]_{\tan}) &=& 0 & \text{on } (0,T_0)\times\Gamma_S, \\
u(0) &=& u_0 & \text{in } \Omega, \\
\tau(0) &=& \tau_0 & \text{in } \Omega.
\end{array}
\right. \tag{2.1}
$$

This system will be described as follows: The unknowns of this system are the velocity field u, the pressure π, and the elastic part of the stress τ. The density of the fluid is denoted by the constant $\rho > 0$, and the symmetric part of the velocity gradient by $Eu = \frac{1}{2}(\nabla u + \nabla u^T)$. The function $\alpha\colon [0,\infty) \to [0,\infty)$ is a given viscosity function and the functions $\mu\colon \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$, $g\colon \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ are given, coupling the elastic part of the stress τ with the velocity field u. The structure conditions

$$\alpha(s) > 0, \quad \alpha(s) + 2s\alpha'(s) > 0, \qquad s \geq 0$$

and, if the domain Ω is unbounded,

$$g(0,0) = 0$$

will play an important role in the analysis of the problem. We consider two kinds of boundary conditions on the disjoint boundary parts Γ_D and Γ_S. On Γ_D, we impose Dirichlet boundary conditions and on Γ_S, we prescribe perfect slip boundary conditions. Furthermore, two initial values, u_0 and τ_0, satisfying the natural compatibility conditions

$$\operatorname{div} u_0 = 0 \quad \text{in } \Omega, \qquad u_0 = 0 \quad \text{on } \Gamma_D, \quad \text{and} \quad (u_0\cdot\nu, [2\alpha(|Eu_0|^2)Eu_0\nu + \mu(\tau_0)\nu]_{\tan}) = 0 \quad \text{on } \Gamma_S \tag{2.2}$$

are given.

The first equation of (2.1) is the balance of momentum, assuming the stress tensor admits the viscoelastic form

$$\mathcal{S}(u,\pi,\tau) = 2\alpha(|Eu|^2)Eu - \pi + \mu(\tau).$$

Since the density ρ is a constant, the second equation characterizes the incompressibility of the fluid. The transport equation describes the evolution of the elastic part of the stress τ. We consider two types of boundary conditions. On the Dirichlet part Γ_D of the boundary, we require no slip boundary conditions, i.e. $u = 0$ on $(0, T_0) \times \Gamma_S$. On the second part of the boundary Γ_S, we impose perfect slip boundary conditions. On this boundary part, the normal part of the velocity $u \cdot \nu$ and the tangential part of the stress in normal direction $[\mathcal{S}(u,\pi,\tau)\nu]_{\tan}$ have to vanish. It should be noted, that the pressure does not appear in this boundary condition, since $[\pi\nu]_{\tan} = 0$.

Problem (2.1) is a generalized Navier-Stokes equation coupled with a transport equation. Since the generalized Stokes equation is parabolic, and the transport equation hyperbolic, we investigate a coupled parabolic-hyperbolic system. If $\Gamma_S \neq \emptyset$, the problem couples not only via the equations given on the domain Ω, but also via the boundary conditions. Even if α is constant the problem is quasilinear, since the unknown velocity field appears in the transport equation in front of the highest derivative of the elastic part of the stress τ.

Similar systems have been intensively studied in the literature. The special case of Oldroyd-B fluids (see Bird, Armstrong and Hassager [BAH87] or Joseph [Jos90] for an overview on viscoelastic fluid models), i.e. for $\Gamma_S = \emptyset$, constant $\alpha > 0$, and setting

$$\mu(\tau) = \mu\tau \quad \text{and} \quad g(\nabla u, \tau) = -\beta\tau + \gamma Eu + \delta((\nabla u)^T\tau + \tau\nabla u), \tag{2.3}$$

with $\mu \in \mathbb{R}$, $\beta \geq 0$, and $\gamma, \delta > 0$ was investigated by Guillaupé and Saut [GS90] in the L_2-setting in bounded domains. They proved the existence of local-in-time strong solutions for large data, as well as global solutions for small data, applying a Schauder fixed point argument. Their method relies on a-priori estimates and compactness arguments.

Later, Fernández-Cara, Guillén and Ortega [FCGO98] proved the existence of a unique strong solution in an L_p-setting similar to our approach for the same model problem as Guillaupé and Saut, also in a bounded domain. They rely on a Schauder fixed point argument as well.

The existence of global weak solutions for the Oldroyd-B model was investigated by Chemin and Masmoudi [CM01]. They replaced the term $(\nabla u)^T\tau + \tau\nabla u$ by the term $Wu\tau - \tau Wu$, where $Wu = \frac{1}{2}(\nabla u - \nabla u^T)$. Lin, Liu, and Zhang [LLZ05], Lei, Liu, and Zhou [LLZ07, LLZ08] as well as Lin and Zhang [LZ08] proved the existence and uniqueness of the Oldroyd-B model, using L_2-methods.

A more general system than Oldroyd-B, where in (2.3) the constant term γ is replaced by a shear-rate dependent function $\gamma(|Eu|^2)$, was investigated, using a modified version of the contraction mapping theorem (Proposition 1.13) in the steady L_2-setting on bounded and exterior domains, by Arada and Sequeira [AS03, AS05]. This model is called generalized Oldroyd-B.

Another generalization of the Oldroyd-B model is the so-called White-Metzner system, where one takes constant $\alpha > 0$, the linear relation $\mu(\tau) = \mu\tau$, $\mu \in \mathbb{R}$, and

$$g(\nabla u, \tau) = \beta(|Eu|^2)\tau + \gamma(|Eu|^2)Eu + \delta((\nabla u)^T\tau + \tau\nabla u)$$

for some functions β and γ. Strong well-posedness of this model in 2D was shown in the L_2-setting by Hakim [Hak94] and later also in 3D by Molinet and Talhouk [MT04] in the non-stationary case in bounded domains.

Also taking into account a nonlinear viscosity term and hence incorporating general shear-thinning and shear-thickening effects, Agranovich and Sobolevskii [AS98] studied a viscoelastic fluid model in the L_2-setting on a bounded domain. However, they replaced in the transport equation the frame-invariant objective derivative

$$\frac{\mathcal{D}_a \tau}{\mathcal{D}t} = \partial_t \tau + u.\nabla\tau - \delta((\nabla u)^T\tau + \tau\nabla u)$$

by a partial derivative ∂_t. This way, one can directly integrate the transport equation and insert the resulting elastic stress into the fluid equation.

Finally, we would like to mention a work by Vorotnikov and Zvyagin [VZ04] who considered (2.1) with $\Omega = \mathbb{R}^n$, $n = 2, 3$, and imposed certain assumptions on g. They proved global existence of unique strong solutions in the L_2-setting, provided that the initial values are sufficiently small and $g(0,0) = 0$. However, due to the L_2-approach using a-priori estimates for a nonlinear system, they impose strong regularity assumptions on the initial data, i.e. $u_0 \in H_2^3(\mathbb{R}^n)$ and $\tau_0 \in H_2^3(\mathbb{R}^n)$.

First, we investigate (2.1) in the case that the domain $\Omega \subset \mathbb{R}^n$ is bounded and the viscosity function α satisfies the structure condition

$$\alpha(s) > 0 \quad \text{and} \quad \alpha(s) + 2s\alpha'(s) > 0, \qquad s \geq 0. \tag{2.4}$$

The main result in this setting is the local-time existence of a unique strong L_p-solution, $n+2 < p < \infty$, for initial values $u_0 \in W^{2-\frac{2}{p}}(\Omega)$ and $\tau_0 \in H_p^1(\Omega)$, satisfying the natural compatibility conditions (2.2) (see Theorem 2.1). We proceed the following way: We linearize the problem and apply Schauder's fixed point theorem to solve the nonlinear problem. To apply Schauder's fixed point theorem (Proposition 1.11), we rely on compact embeddings of Sobolev spaces, and therefore on the boundedness of the domain. The uniqueness of the solution is shown with an energy argument.

Next, we focus on (2.1) in the case of an unbounded domain Ω. More precisely we consider domains, such that for any $n < q < \infty$ the Helmholtz decomposition exists for $L_r(\Omega)$, $r \in \{q, \frac{q}{q-1}\}$, and for any $\lambda \geq 0$, a shift of the Stokes operator $\lambda + A_r$, $r \in \{q, \frac{q}{q-1}\}$, admits bounded imaginary powers with a power angle less than $\frac{\pi}{2}$. Examples of such domains are exterior domains, layers, half spaces, and the whole space. In this setting, we have the additional assumptions that the slip part of the boundary is empty ($\Gamma_S = \emptyset$), $\alpha > 0$ is constant, and $g(0,0) = 0$. In the bounded domain case, we relied on compact embeddings. These embeddings do not hold in unbounded domains. Instead of Schauder's fixed point theorem, we use now a modified version of the contraction mapping principle (Proposition 1.13). In order to apply this proposition, suitable estimates of the linearized problem are needed. To proof this estimates, we use the fact, that a shift of Stokes operator admits bounded imaginary powers. This is the main reason for considering the equation with a constant viscosity function. Only if the viscosity function is constant, the associated linearization of (2.1) is connected to the Stokes problem. The result is local-in-time well posedness in the $L_p - L_q$-setting, $\frac{1}{p} + \frac{n}{2q} < \frac{1}{2}$, for initial values $(u_0, \tau_0) \in (L_{q,\sigma}(\Omega), D(A_q))_{1-\frac{1}{p},p} \times H_q^1(\Omega)$ (see Theorem 2.6). In the case of an Oldroyd-B fluid, where in addition the special form (2.3) of g and μ is assumed, we extend this result to a more value of p and q. More precisely, we can prove the same result for $1 < p < \infty$ with $p \neq 2$ and $n < q < \infty$ (see Theorem 2.7).

Lastly, we investigate (2.1) in the case that $\Omega = \mathbb{R}^n_+$ is a half space. Similar to Theorem 2.6, we assume here that the viscosity function is constant and $g(0,0) = 0$. In contrast to Theorem 2.6, we consider here perfect slip boundary conditions ($\Gamma_S \neq \emptyset$). The idea is to apply a modified version of the contraction mapping principle (Proposition 1.13) to the associated linearization. Once more, a suitable estimate of the linearized problem is needed. In the proof of Theorem 2.6, we consider (2.1) with homogeneous Dirichlet boundary conditions and can therefore use the bounded imaginary powers of the Stokes operator to proof this estimate. The boundary conditions of the linearization are not homogeneous in the case of perfect slip boundary condition and thus, we cannot apply this method directly. But, in the case $\Omega = \mathbb{R}^n_+$, we can use an explicit solution formula of the Stokes problem to deduce the required estimate.

2.1 Generalized viscoelastic fluids on bounded domains

In this section, $\Omega \subset \mathbb{R}^n$ is always a bounded domain with a $C^{2,1}$-boundary and $n+2 < p < \infty$. The outer normal on $\partial\Omega$ is always denoted by ν. The aim of this section is to establish local-in-time solvability of (2.1). More precisely, for the given external force $f \in L_p(0, T_0; L_p(\Omega))$ as well as given initial values $u_0 \in W_p^{2-\frac{2}{p}}(\Omega)$ and $\tau_0 \in H_p^1(\Omega)$, satisfying the compatibility condition (2.2), we prove the existence of a small time T, $0 < T < T_0$, and of a unique strong solution of (2.1) on the time interval $(0, T)$ in the regularity class

$$\begin{aligned} &u \in H_p^1(0,T;L_p(\Omega)) \cap L_p(0,T;H_p^2(\Omega)), \quad \pi \in L_p(0,T;\widehat{H}_p^1(\Omega)),\\ &\text{and} \quad \tau \in W_\infty^1(0,T;L_p(\Omega)) \cap L_\infty(0,T;H_p^1(\Omega)). \end{aligned}$$

Let us state the main result of this section:

Theorem 2.1. *Fix $n \in \mathbb{N}$, $n \geq 2$, $n+2 < p < \infty$, and $T_0, \rho > 0$. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class $C^{2,1}$, such that the boundary $\partial\Omega = \Gamma_D \cup \Gamma_S$ decomposes into two disjoint subsets, Γ_D and Γ_S, which are open and closed in $\partial\Omega$. Furthermore, we assume $\mu \in C^3(\mathbb{R}^{n\times n}, \mathbb{R}^{n\times n})$, $g \in C^1(\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}, \mathbb{R}^{n\times n})$, and that $\alpha \in C^{1,1}([0,\infty))$ satisfies the structure condition*

$$\alpha(s) > 0 \quad \text{and} \quad \alpha(s) + 2s\alpha'(s) > 0, \qquad s \geq 0.$$

Then, for each $f \in L_p(0,T_0;L_p(\Omega))$, $u_0 \in W_p^{2-\frac{2}{p}}(\Omega)$, and $\tau_0 \in H_p^1(\Omega)$, satisfying the compatibility conditions

$$\operatorname{div} u_0 = 0 \quad \text{in } \Omega, \qquad u_0 = 0 \quad \text{on } \Gamma_D, \quad \text{and} \quad (u_0 \cdot \nu, [2\alpha(|Eu_0|^2)Eu_0\nu + \mu(\tau_0)\nu]_{tan}) = 0 \quad \text{on } \Gamma_S,$$

there exists a time $0 < T < T_0$ and a unique strong solution (u, π, τ) of (2.1) on the time interval $(0, T)$ in the regularity class

$$\begin{aligned} &u \in H_p^1(0,T;L_p(\Omega)) \cap L_p(0,T;H_p^2(\Omega)), \quad \pi \in L_p(0,T;\widehat{H}_p^1(\Omega)),\\ &\text{and} \quad \tau \in W_\infty^1(0,T;L_p(\Omega)) \cap L_\infty(0,T;H_p^1(\Omega)). \end{aligned}$$

For a corresponding small data result, we refer the reader to [GGN12].

Remark 2.2. In the situation of Theorem 2.1, we can weaken the differentiability assumption on μ, if we consider only Dirichlet boundary condition ($\Gamma_S = \emptyset$). In this case $\mu \in C^1(\mathbb{R}^{n\times n}, \mathbb{R}^{n\times n})$ is sufficient.

Sketch of the proof

Let us first outline the proof. Since we are interested in arbitrarily large initial data, it is convenient to reduce (2.1) to $u_0 = 0$. In this case, the embedding constants in the proposition on embedding theorems (Proposition 1.14) are independent of T, $0 < T < T_0$. For this purpose, we introduce a function u_*, incorporating the initial condition u_0, the external force f, and the compatibility condition (2.2). In the subsection on the generalized Stokes problem (Subsection 1.2.3), we introduced a quasilinear operator

$$\mathcal{A}(E\tilde{u})\tilde{v} \quad \text{with} \quad \mathcal{A}(E\tilde{v})\tilde{v} = -\operatorname{Div} 2\alpha(|E\tilde{v}|^2)E\tilde{v},$$

and the corresponding slip boundary operator $\mathcal{B}_{\mathcal{S}}(E\tilde{u})$. Setting $u = w + u_*$, we can formulate (2.1) in the equivalent form

$$\begin{cases} \rho\partial_t w + \mathcal{A}(Eu_*)w + \nabla\psi &= f_* + F_w(w) + F_\tau(\tau) & \text{in } (0,T_0)\times\Omega, \\ \operatorname{div} w &= 0 & \text{in } (0,T_0)\times\Omega, \\ \partial_t\tau + (w+u_*).\nabla\tau &= G(w,\tau) & \text{in } (0,T_0)\times\Omega, \\ w &= 0 & \text{on } (0,T_0)\times\Gamma_D, \\ (w\cdot\nu, \mathcal{B}_{\mathcal{S}}(Eu_*)w) &= (0, h_* + H_w(w) + H_\tau(\tau)) & \text{on } (0,T_0)\times\Gamma_S, \\ w(0) &= 0 & \text{in } \Omega, \\ \tau(0) &= \tau_0 & \text{in } \Omega, \end{cases}$$

where $\mathcal{A}(Eu_*)$ is now a fixed operator, $\mathcal{B}_{\mathcal{S}}(Eu_*)$ is the corresponding slip boundary operator, f_*, h_* are suitable functions, and $F_w, F_\tau, G, H_w, H_\tau$ are given nonlinearities, which are calculated below.

In the next step, we rewrite the previous equation equivalently in form of a fixed point equation in a suitable Banach space. The fixed point map is defined by $\Phi\colon (\tilde{w},\tilde{\tau}) \mapsto (w,\tau)$, where (w,τ) is the solution of

$$\begin{cases} \rho\partial_t w + \mathcal{A}(Eu_*)w + \nabla\psi &= f_* + F_w(\tilde{w}) + F_\tau(\tilde{\tau}) & \text{in } (0,T_0)\times\Omega, \\ \operatorname{div} w &= 0 & \text{in } (0,T_0)\times\Omega, \\ \partial_t\tau + (\tilde{w}+u_*).\nabla\tau &= G(\tilde{w},\tilde{\tau}) & \text{in } (0,T_0)\times\Omega, \\ w &= 0 & \text{on } (0,T_0)\times\Gamma_D, \\ (w\cdot\nu, \mathcal{B}_{\mathcal{S}}(Eu_*)w) &= (0, h_* + H_w(\tilde{w}) + H_\tau(\tilde{\tau})) & \text{on } (0,T_0)\times\Gamma_S, \\ w(0) &= 0 & \text{in } \Omega, \\ \tau(0) &= \tau_0 & \text{in } \Omega. \end{cases}$$

The task is to find a fixed point of Φ. It is worth pointing out that the above equations are decoupled. We have to solve a generalized Stokes equation with Dirichlet and inhomogeneous perfect slip boundary conditions and a transport equation. Maximal regularity of generalized Stokes problems of this kind were developed by Bothe and Prüß [BP07] (see Proposition 1.8) and the transport equation is investigated in Proposition 1.10. By Schauder's fixed point theorem, we can prove the existence of a fixed point.

It remains to show the uniqueness of the solution. Since the time interval $(0,T)$ and the domain Ω are bounded, and $p > n+2$, the solutions in the above mentioned regularity class also belong to L_2. This enables us to apply an energy method to show the uniqueness of the solution. The principle point in the proof of the energy estimate of the solution is, that the strong monotonicity of the operator $X \mapsto 2\alpha(|X|^2)X$ is implied by the structure condition (2.4).

Proof of Theorem 2.1. The proof is divided in an existence part and a uniqueness part.

Existence with Schauder's fixed point theorem

The first step is to reduce (2.1) to $u_0 = 0$ and $f = 0$.

Reduction to $u_0 = 0$ and $f = 0$

We construct a function $u_* \in H^1_p(0,T_0;L_p(\Omega)) \cap L_p(0,T_0;H^2_p(\Omega))$ with $u_*(0) = u_0$ as the solution of a Stokes problem. In order to satisfy the compatibility conditions required in Proposition 1.8, we define

$$h = [\mathcal{E}_t 2(\alpha(0) - \alpha(|Eu_0|^2))Eu_0\nu]_{\tan} \in W_p^{\frac{1}{2}-\frac{1}{2p}}(0,T_0;L_p(\Gamma_S)) \cap L_p(0,T_0;W_p^{1-\frac{1}{p}}(\Gamma_S)),$$

where $\mathcal{E}_t\colon W^{1-\frac{3}{p}}(\Gamma_S) \to W_p^{\frac{1}{2}-\frac{1}{2p}}(0,T_0;L_p(\Gamma_S)) \cap L_p(0,T_0;W_p^{1-\frac{1}{p}}(\Gamma_S))$ is the extension operator given by Proposition 1.15. It should be noted that

$$\mu(\tau_0) \in W^{1-\frac{1}{p}}(\Gamma_S) \quad \text{and} \quad (\alpha(|Eu_0|^2) - \alpha(0))Eu_0 \in W^{1-\frac{3}{p}}(\Gamma_S),$$

due to the proposition on pointwise multiplications (Proposition 1.16) and the proposition on Nemytskij operators (Proposition 1.17). Then, by construction

$$2\alpha(0)[Eu_0\nu]_{\tan} = h(0) - [\mu(\tau_0)\nu]_{\tan} \quad \text{thanks to} \quad [2\alpha(|Eu_0|^2)Eu_0\nu + \mu(\tau_0)\nu]_{\tan} = 0,$$

and hence, there exists a unique solution

$$(u_*,\pi_*) \in H^1_p(0,T;L_p(\Omega)) \cap L_p(0,T;H^2_p(\Omega)) \times L_p(0,T;\widehat{H}^1_p(\Omega))$$

of the Stokes problem

$$\tag{2.5} \left\{ \begin{array}{rcll} \rho\partial_t u_* - \alpha(0)\Delta u_* + \nabla\pi_* &=& f & \text{in } (0,T_0)\times\Omega, \\ \operatorname{div} u_* &=& 0 & \text{in } (0,T_0)\times\Omega, \\ u_* &=& 0 & \text{on } (0,T_0)\times\Gamma_D, \\ (u_*\cdot\nu, 2[\alpha(0)Eu_*\nu]_{\tan}) &=& (0, h - [\mu(\tau_0)\nu]_{\tan}) & \text{on } (0,T_0)\times\Gamma_S, \\ u_*(0) &=& u_0 & \text{in } \Omega, \end{array} \right.$$

by Proposition 1.8. We set

$$(u,\pi) = (w + u_*, \psi + \pi_*).$$

Then, (u,π,τ) solves (2.1) if and only if (w,ψ,τ) solves

$$\tag{2.6} \left\{ \begin{array}{rcll} \rho\partial_t w + \mathcal{A}(Eu_*)w + \nabla\psi &=& f_* + F_w(w) + F_\tau(\tau) & \text{in } (0,T_0)\times\Omega, \\ \operatorname{div} w &=& 0 & \text{in } (0,T_0)\times\Omega, \\ \partial_t\tau + (w+u_*).\nabla\tau &=& G(w,\tau) & \text{in } (0,T_0)\times\Omega, \\ w &=& 0 & \text{on } (0,T_0)\times\Gamma_D, \\ (w\cdot\nu, \mathcal{B}_{\mathcal{S}}(Eu_*)w) &=& (0, h_* + H_w(w) + H_\tau(\tau)) & \text{on } (0,T_0)\times\Gamma_S, \\ w(0) &=& 0 & \text{in } \Omega, \\ \tau(0) &=& \tau_0 & \text{in } \Omega, \end{array} \right.$$

where the fixed operator $\mathcal{A}(Eu_*)$ and its corresponding perfect slip boundary operator $\mathcal{B}_{\mathcal{S}}(Eu_*)$ are defined as in the subsection on the generalized Stokes equation (Subsection 1.2.3) and the function and nonlinearities f_*, F_w, F_τ, $G(w,\tau)$, h_*, $H_w(w)$, and $H_\tau(\tau)$ are defined and investigated below. First, we write the equation

$$\rho(\partial_t v + v.\nabla v) - \operatorname{Div} 2\alpha(|Ev|^2)Ev + \nabla\theta = \operatorname{Div}\mu(\tau) + f \quad \text{in } (0,T_0)\times\Omega$$

in the form

$$\rho\partial_t w + \mathcal{A}(Eu_*)w + \nabla\psi = f_* + F_w(w) + F_\tau(\tau) \quad \text{in } (0,T_0)\times\Omega,$$

where the function f_* is given by

$$f_* := -\rho\partial_t u_* - \nabla\pi_* + f - \mathcal{A}(Eu_*)u_* - \rho u_*.\nabla u_* + \operatorname{Div}\mu(\tau_0),$$

the nonlinearity F_w and F_τ on the right-hand side of the generalized Stokes equation are defined by

$$F_w(w) := (\mathcal{A}(Eu_*) - \mathcal{A}(E(w+u_*)))(w+u_*) - \rho u_*.\nabla w - \rho w.\nabla u_* - \rho w.\nabla w,$$

and

$$F_\tau(\tau) := \operatorname{Div}(\mu(\tau) - \mu(\tau_0)),$$

where we used the identity $\mathcal{A}(E(w+u_*))(w+u_*) = -\operatorname{Div}\alpha(|E(w+u_*)|^2)E(w+u_*)$ (see (1.4)). Further, by (2.5), we simplify

$$f_* = -\alpha(0)\Delta u_* - \mathcal{A}(Eu_*)u_* - \rho u_*.\nabla u_* + \operatorname{Div}\mu(\tau_0) \in L_p(0,T_0;L_p(\Omega)).$$

The right-hand side G of the transport equation is of the form

$$G(w,\tau) = g(\nabla(w+u_*),\tau).$$

To deduce a suitable representation of the slip boundary condition, we use the identity (1.6), i.e.

$$\mathcal{B}_{\mathcal{S}}(Eu_*)w = 2\alpha(|Eu_*|^2)[Ew\nu]_{\tan} + 4\alpha'(|Eu_*|^2)(Eu_* : Ew)[Eu_*\nu]_{\tan}.$$

Therefore, the boundary condition on the slip part of the boundary

$$[2\alpha(|Eu|^2)Eu\nu + \mu(\tau)\nu]_{\tan} = 0 \quad \text{on } (0,T_0)\times\Gamma_S$$

can be written equivalently in the form

$$\begin{aligned}
0 &= [2\alpha(|E(w+u_*)|^2)E(w+u_*)\nu + \mu(\tau)\nu]_{\tan} \\
&= [2\alpha(|E(w+u_*)|^2)E(w+u_*)\nu + \mu(\tau)\nu]_{\tan} - \mathcal{B}_{\mathcal{S}}(Eu_*)w + \mathcal{B}_{\mathcal{S}}(Eu_*)w \\
&= 2\big[\alpha(|E(w+u_*)|^2)E(w+u_*)\nu - \alpha(|Eu_*|^2)Ew\nu - 2\alpha'(|Eu_*|^2)(Eu_* : Ew)Eu_*\nu\big]_{\tan} \\
&\quad + \big[\mu(\tau)\nu\big]_{\tan} + \mathcal{B}_{\mathcal{S}}(Eu_*)w \\
&= 2\big[\alpha(|E(w+u_*)|^2)E(w+u_*)\nu - \alpha(|Eu_*|^2)E(w+u_*)\nu - 2\alpha'(|Eu_*|^2)(Eu_* : Ew)Eu_*\nu\big]_{\tan} \\
&\quad + \big[(\mu(\tau)-\mu(\tau_0))\nu\big]_{\tan} + \big[\mu(\tau_0)\nu + \alpha(|Eu_*|^2)Eu_*\nu\big]_{\tan} + \mathcal{B}_{\mathcal{S}}(Eu_*)w
\end{aligned}$$

and this is equivalent to

$$\mathcal{B}_{\mathcal{S}}(Eu_*)w = h_* + H_w(w) + H_\tau(w)$$

with the function

$$h_* := -\big[2\alpha(|Eu_*|^2)Eu_*\nu + \mu(\tau_0)\nu\big]_{\tan} \in {}_0W_p^{\frac{1}{2}-\frac{1}{2}}((0,T_0);L_p(\Gamma_S)) \cap L_p((0,T_0);W^{1-\frac{1}{p}}(\Gamma_S)),$$

and the nonlinear terms

$$\begin{aligned} H_w(w) := &-2\big(\alpha(|E(w+u_*)|^2) - \alpha(|Eu_*|^2) - 2\alpha'(|Eu_*|^2)(Eu_* : Ew)\big)\big[Eu_*\nu\big]_{\tan} \\ &- 2\big(\alpha(|E(w+u_*)|^2) - \alpha(|Eu_*|^2)\big)\big[Ew\nu\big]_{\tan}, \end{aligned}$$

and

$$H_\tau(\tau) := -\big[(\mu(\tau) - \mu(\tau_0))\nu\big]_{\tan}.$$

It is worth pointing out that $h_*(0) = 0$ by $u_*(0) = u_0$ and the compatibility conditions.

Fixed point formulation

The aim is to formulate (2.6) in form of a fixed point problem in a suitable Banach space. Let $n+2 < p < r < \infty$. We define of the solution spaces for the velocity field ${}_0\mathbb{E}_u(T,\Omega)$ and define the solution space for the elastic part of the stress $\mathbb{E}_\tau(T,\Omega)$ via

$$\begin{aligned} {}_0\mathbb{E}_{u,c}(T,\Omega) &:= \{w \in {}_0\mathbb{E}_u(T,\Omega) = {}_0H_p^1(0,T;L_p(\Omega)) \cap L_p(0,T;H_p^2(\Omega))\colon w\cdot\nu = 0 \text{ on } \partial\Omega\}, \\ \mathbb{E}_\tau(T,\Omega) &:= \widehat{H}_r^1(0,T;L_p(\Omega)) \cap L_\infty(0,T;H_p^1(\Omega)), \end{aligned}$$

Moreover, we recall the definition of space for the velocity field, where we do not prescribe the initial value

$$\mathbb{E}_u(T,\Omega) = H_p^1(0,T;L_p(\Omega)) \cap L_p(0,T;H_p^2(\Omega)).$$

Further, we define the space for the right hand side of the generalized Stokes equation $\mathbb{F}_f(T,\Omega_0)$ as well as the right-hand side of the transport equation $\mathbb{G}(T,\Omega)$ and recall the definition of the space for the right-hand side of the boundary condition ${}_0\mathbb{H}_u(T,\Gamma_S)$:

$$\begin{aligned} \mathbb{F}_f(T,\Omega) &:= L_p(0,T;L_p(\Omega)), \\ \mathbb{G}(T,\Omega) &:= L_r(0,T;L_p(\Omega)) \cap L_1(0,T;H_p^1(\Omega)), \\ {}_0\mathbb{H}_u(T,\Gamma_S) &= {}_0W_p^{\frac{1}{2}-\frac{1}{2p}}(0,T;L_p(\Gamma_S)) \cap L_p(0,T;W_p^{1-\frac{1}{p}}(\Gamma_S)). \end{aligned}$$

It should be noted that the definition of ${}_0\mathbb{E}_u(T,\Omega)$, $\mathbb{E}_u(T,\Omega)$, and ${}_0\mathbb{H}_u(T,\Gamma_S)$ corresponds to the definition in the preliminaries. Problem (2.6) can be rewritten equivalently as a fixed point problem of the map

(2.7)
$$\begin{aligned} \Phi\colon {}_0\mathbb{E}_{u,c}(T,\Omega)\times\mathbb{E}_\tau(T,\Omega) &\to {}_0\mathbb{E}_{u,c}(T,\Omega)\times\mathbb{E}_\tau(T,\Omega), \\ (w,\tau) &\mapsto \tilde{\Phi}_{0,\tau_0}(w, f_* + F_w(w) + F_\tau(\tau), G(w,\tau), h_* + H_w(w) + H_\tau(\tau)), \end{aligned}$$

where

$$\tilde{\Phi}_{0,\tau_0}\colon {}_0\mathbb{E}_{u,c}(T,\Omega)\times\mathbb{F}_f(T,\Omega)\times\mathbb{G}(T,\Omega)\times{}_0\mathbb{H}_u(T,\Gamma_S) \to {}_0\mathbb{E}_{u,c}(T,\Omega)\times\mathbb{E}_\tau(T,\Omega),$$
$$(\tilde{w},\tilde{f},\tilde{g},\tilde{h}) \mapsto (w,\tau)$$

denotes the solution operator to the problem

$$\left\{\begin{array}{rcll} \rho\partial_t w+\mathcal{A}(Eu_*)w+\nabla\psi &=& \tilde{f} & \text{in } (0,T_0)\times\Omega,\\ \operatorname{div} w &=& 0 & \text{in } (0,T_0)\times\Omega,\\ \partial_t\tau+(\tilde{w}+u_*).\nabla\tau &=& \tilde{g} & \text{in } (0,T_0)\times\Omega,\\ w &=& 0 & \text{on } (0,T_0)\times\Gamma_D,\\ (w\cdot\nu,\mathcal{B}_{\mathcal{S}}(Eu_*)w) &=& (0,[\tilde{h}]_{\tan}) & \text{on } (0,T_0)\times\Gamma_S,\\ w(0) &=& 0 & \text{in } \Omega,\\ \tau(0) &=& \tau_0 & \text{in } \Omega. \end{array}\right. \tag{2.8}$$

It should be noted that Φ and $\tilde{\Phi}_{0,\tau_0}$ are well-defined: The generalized Stokes equation and the transport equation are decoupled in (2.8) and can be solved separately. Since $w(0)=0$ and $\tilde{h}(0)=0$, the compatibility conditions in the result on solvability of the generalized Stokes equation (Proposition 1.8) are fulfilled, and thus, there exists a unique solution of the generalized Stokes equation. By the proposition on the transport equation (Proposition 1.16), we can also solve the transport equation. This shows that $\tilde{\Phi}_{0,\tau_0}$ is well-defined. In Lemma 2.3 we show the mapping properties

$$F_w\colon {}_0\mathbb{E}_{u,c}(T,\Omega)\to\mathbb{F}_f(T,\Omega),\quad F_\tau\colon \mathbb{E}_\tau(T,\Omega)\to\mathbb{F}_f(T,\Omega),\quad G\colon {}_0\mathbb{E}_{u,c}(T,\Omega)\times\mathbb{E}_\tau(T,\Omega)\to\mathbb{G}(T,\Omega),$$
$$H_w\colon {}_0\mathbb{E}_{u,c}(T,\Omega)\to{}_0\mathbb{H}_u(T,\Gamma_S)\quad\text{and}\quad H_\tau\colon {}_0\mathbb{E}_\tau(T,\Omega)\to{}_0\mathbb{H}_u(T,\Gamma_S).$$

This implies that Φ is well-defined.

Analysis of Φ

Next, we show that Φ admits a fixed point. For this purpose, we define for $0<R_1,R_2,R_3<\infty$, $0<T<T_0$, and $n+2<p<r<\infty$ the closed and convex sets

$$\begin{array}{lcl} \mathcal{K}_w(T,R_1) &:=& \{w\in{}_0\mathbb{E}_{u,c}(T,\Omega)\colon \|w\|_{\mathbb{E}_u(T,\Omega)}\le R_1\},\\ \mathcal{K}_\tau(T,R_2,R_3) &:=& \{\tau\in\mathbb{E}_\tau(T,\Omega)\colon \tau(0)=\tau_0,\ \|\tau\|_{L_\infty(0,T;H^1_p(\Omega))}\le R_2 \text{ and } \|\partial_t\tau\|_{T,\Omega,r,p}\le R_3\},\\ \mathcal{K}(T,R_1,R_2,R_3) &:=& \mathcal{K}_w(T,R_1)\times\mathcal{K}_\tau(T,R_2,R_3). \end{array}$$

The map Φ maps $\mathcal{K}(T,R_1,R_2,R_3))$ into itself

We show, that we can choose R_1, R_2, R_3, and $0<T<T_0$, such that

$$\Phi(\mathcal{K}(T,R_1,R_2,R_3))\subset\mathcal{K}(T,R_1,R_2,R_3).$$

To show this relation, we estimate the nonlinearities in the following lemma.

Lemma 2.3. *Fix $n\in\mathbb{N}$, $n\ge 2$, $T_0,R_0,R_2,R_3>0$, and $n+2<p<r<\infty$. Let $\Omega\subset\mathbb{R}^n$ be a bounded domain of class $C^{2,1}$, such that the boundary $\partial\Omega=\Gamma_S\cup\Gamma_D$ decomposes into two disjoint subsets, Γ_D and Γ_S, which are open and closed in $\partial\Omega$. In addition, we assume that $\alpha\in C^{1,1}([0,\infty))$ satisfies the structure condition (2.4), $\mu\in C^3(\mathbb{R}^{n\times n},\mathbb{R}^{n\times n})$ and $g\in C^1(\mathbb{R}^{n\times n}\times\mathbb{R}^{n\times n},\mathbb{R}^{n\times n})$. Then,* *there exists a constant $C > 0$ and a function $O\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $O(t) \to 0$ for $t \to 0$, such that for all $R_1 \in (0, R_0)$, $T \in (0, T_0)$ and $(w, \tau) \in \mathcal{K}(T, R_1, R_2, R_3)$ the estimates*

$$\begin{array}{lcl}
\|F_w(w)\|_{\mathbb{F}_f(T,\Omega)} & \leq & CR_1^2 + O(T), \\
\|F_\tau(\tau)\|_{\mathbb{F}_f(T,\Omega)} & \leq & O(T), \\
\|G(w,\tau)\|_{\mathbb{G}(T,\Omega)} & \leq & O(T), \\
\|G(w,\tau)\|_{T,\Omega,\infty,p} & \leq & C, \\
\|H_w(w)\|_{{}_0\mathbb{H}_u(T,\Gamma_S)} & \leq & CR_1^2 + O(T), \\
\|H_\tau(\tau)\|_{{}_0\mathbb{H}(T,\Gamma_S)} & \leq & O(T)
\end{array}$$

hold.

Proof. Let $0 < R_0, R_2, R_3, T_0$, $0 < R_1 < R_0$, $0 < T < T_0$, and $(w, \tau) \in \mathcal{K}(T, R_1, R_2, R_3)$. We denote by C a generic constant and by $O\colon \mathbb{R}_+ \to \mathbb{R}_+$ a generic function, with $O(t) \to 0$ for $t \to 0$, which may change from line to line, but are always independent of T, R_1, w, and τ.

The terms F_w and H_w were already investigated by Bothe and Prüß [BP07, Section 9]. They proved the required estimates

$$\begin{array}{lcl}
\|F_w(w)\|_{\mathbb{F}_f(T,\Omega)} & \leq & CR_1^2 + O(T), \\
\|H_w(w)\|_{{}_0\mathbb{H}_u(T,\Gamma_S)} & \leq & CR_1^2 + O(T).
\end{array}$$

We analyse the remaining terms F_τ, G, and H_τ. By the proposition on embedding theorems (Proposition 1.14), there exists a constant C_* with

$$\|w\|_{L_\infty(0,T;W^1_\infty(\Omega))} + \|u_*\|_{L_\infty(0,T;W^1_\infty(\Omega))} + \|\tau\|_{L_\infty(0,T;L_\infty(\Omega))} \leq C_*.$$

First, we investigate F_τ. By the chain rule, we deduce that

$$\begin{aligned}
\|F_\tau(\tau)\|_{T,\Omega,p,p} &= \|\operatorname{Div}(\mu(\tau) - \mu(\tau_0))\|_{T,\Omega,p,p} \\
&\leq \sup_{|\eta| \leq C_*} |(\nabla\mu)(\eta)| (\|\nabla\tau\|_{T,\Omega,p,p} + \|\nabla\tau_0\|_{T,\Omega,p,p}) \\
&\leq \sup_{|\eta| \leq C_*} |(\nabla\mu)(\eta)| T^{\frac{1}{p}} (\|\nabla\tau\|_{T,\Omega,\infty,p} + \|\nabla\tau_0\|_{T,\Omega,\infty,p}) \\
&\leq \sup_{|\eta| \leq C_*} |(\nabla\mu)(\eta)| T^{\frac{1}{p}} (R_2 + \|\tau_0\|_{H^1_p(\Omega)}) \\
&\leq O(T).
\end{aligned}$$

Next, we estimate G. For $r' \in \{1, r, \infty\}$, we conclude that

$$\begin{aligned}
\|G(w,\tau)\|_{T,\Omega,r',p} = \|g(\nabla(w + u_*), \tau)\|_{T,\Omega,r',p} &\leq T^{\frac{1}{r'}} |\Omega|^{\frac{1}{p}} \|g(\nabla(w + u_*), \tau)\|_{T,\Omega,\infty,\infty} \\
&\leq \sup_{|\eta_1|,|\eta_2| < C_*} T^{\frac{1}{r'}} |\Omega|^{\frac{1}{p}} |g(\eta_1, \eta_2)|.
\end{aligned}$$

It follows that

$$\|G(w,\tau)\|_{T,\Omega,1,p} \leq O(T), \quad \|G(w,\tau)\|_{T,\Omega,r,p} \leq O(T), \quad \text{and} \quad \|G(w,\tau)\|_{T,\Omega,\infty,p} \leq C.$$

Moreover, we analyse the spatial derivative of G. By the chain rule, it may be concluded that

$$
\begin{aligned}
&\|\nabla G(w,\tau)\|_{T,\Omega,1,p} \\
&\quad\le \sup_{|\eta_1|,|\eta_2|<C_*} |(\nabla g)(\eta_1,\eta_2)|\big(\|\nabla^2(w+u_*)\|_{T,\Omega,1,p} + \|\nabla\tau\|_{T,\Omega,1,p}\big) \\
&\quad\le \sup_{|\eta_1|,|\eta_2|<C_*} |(\nabla g)(\eta_1,\eta_2)|\big(T^{1-\frac{1}{p}}\|w\|_{{}_0\mathbb{E}_u(T,\Omega)} + T^{1-\frac{1}{p}}\|u_*\|_{\mathbb{E}_u(T,\Omega)} + T\|\tau\|_{L_\infty(0,T;H^1_p(\Omega))}\big) \\
&\quad\le \sup_{|\eta_1|,|\eta_2|<C_*} |(\nabla g)(\eta_1,\eta_2)|\big(T^{1-\frac{1}{p}}R_1 + T^{1-\frac{1}{p}}\|u_*\|_{{}_0\mathbb{E}_u(T,\Omega)} + TR_2\big) \\
&\quad\le O(T).
\end{aligned}
$$

The analysis of the nonlinear term $H_\tau(\tau) = -[\mu(\tau)-\mu(\tau_0)]_{\tan}$ remains. Applying Proposition 1.19 yields

$$\|H_\tau(\tau)\|_{{}_0\mathbb{H}_u(T,\Gamma_S)} \le C\|\tau-\tau_0\|_{{}_0\mathbb{H}_u(T,\Gamma_S)}.$$

By the continuity of the trace operator (see Proposition 1.15)

$$\gamma_{\Gamma_S}\colon {}_0H^{\frac{1}{2}}_p(0,T;L_p(\Omega)) \cap L_p((0,T;H^1_p(\Omega)) \to {}_0\mathbb{H}_u(T,\Gamma_S),$$

it follows that

$$
\begin{aligned}
\|H_\tau(\tau)\|_{{}_0\mathbb{H}_u(T,\Gamma_S)} &\le C\|\tau-\tau_0\|_{{}_0H^{\frac{1}{2}}_p(0,T;L_p(\Omega))\cap L_p((0,T;H^1_p(\Omega))} \\
&\le C(\|\tau-\tau_0\|_{{}_0H^1_p(0,T;L_p(\Omega))} + \|\tau-\tau_0\|_{L_p(0,T;H^1_p(\Omega))}) \\
&\le O(T)(\|\tau-\tau_0\|_{{}_0H^1_r(0,T;L_p(\Omega))} + \|\tau-\tau_0\|_{L_\infty(0,T;H^1_p(\Omega))}) \\
&\le O(T)(\|\partial_t\tau\|_{T,\Omega,r,p} + \|\tau\|_{L_\infty(0,T;H^1_p(\Omega))} + \|\tau_0\|_{H^1_p(\Omega)}) \\
&\le O(T)(R_2+R_3+\|\tau_0\|_{H^1_p(\Omega)}) \\
&\le O(T).
\end{aligned}
$$

This completes the proof. □

Remark 2.4. In the situation of Lemma 2.3, we can weaken the differentiability assumption on μ, if we assume that $\Gamma_S = \emptyset$. In this case, $\mu \in C^1(\mathbb{R}^{n\times n},\mathbb{R}^{n\times n})$ is sufficient, since we only need the higher regularity to estimate H_τ.

We are now in a position to show, that we can choose $0 < T_0, R_0, R_2, R_3$, $0 < T < T_0$, and $0 < R_1 < R_0$, such that

$$\Phi(\mathcal{K}(T,R_1,R_2,R_3)) \subset \mathcal{K}(T,R_1,R_2,R_3).$$

Let $(w,\tau) = \Phi(\tilde{w},\tilde{\tau})$ with $(\tilde{w},\tilde{\tau}) \in \mathcal{K}(T,R_1,R_2,R_3)$. By the solvability result on the generalized Stokes problem (Proposition 1.8), the proposition on the transport equation (Proposition 1.10), and the previous lemma on the nonlinearities, it follows that

$$
\begin{aligned}
&\|w\|_{{}_0\mathbb{E}_u(T,\Omega)}\\
&\quad= \big\|\big(\tilde\Phi_{0,\tau_0}(\tilde w, f_* + F_w(\tilde w) + F_\tau(\tilde\tau), G(\tilde w,\tilde\tau), h_* + H_w(\tilde w) + H_\tau(\tilde\tau))\big)_1\big\|_{{}_0\mathbb{E}_u(T,\Omega)}\\
&\quad\le C\big(\|f_* + F_w(\tilde w) + F_\tau(\tilde\tau)\|_{\mathbb{F}_f(T,\Omega)} + \|h_* + H_w(\tilde w) + H_\tau(\tilde\tau)\|_{{}_0\mathbb{H}_u(T,\Gamma_S)}\big)\\
&\quad\le C\big(\|f_*\|_{\mathbb{F}_f(T,\Omega)} + \|F_w(\tilde w)\|_{\mathbb{F}_f(T,\Omega)} + \|F_\tau(\tilde\tau)\|_{\mathbb{F}_f(T,\Omega)} + \|h_*\|_{{}_0\mathbb{H}_u(T,\Gamma_S)} + \|H_w(\tilde w)\|_{{}_0\mathbb{H}_u(T,\Gamma_S)}\\
&\qquad + \|H_\tau(\tilde\tau)\|_{{}_0\mathbb{H}_u(T,\Gamma_S)}\big)\\
&\quad\le CR_1^2 + O(T),
\end{aligned}
$$

and

$$
\begin{aligned}
&\|\tau\|_{L_\infty(0,T;H^1_p(\Omega))}\\
&\quad= \|\big(\tilde\Phi_{0,\tau_0}(\tilde w, f_* + F_w(\tilde w) + F_\tau(\tilde\tau), G(\tilde w,\tilde\tau), h_* + H_w(\tilde w) + H_\tau(\tilde\tau))\big)_2\|_{L_\infty(0,T;H^1_p(\Omega))}\\
&\quad\le C^{(1)}_{\mathrm{tra}}\big(\|\tau_0\|_{H^1_p(\Omega)} + \|G(\tilde w,\tilde\tau)\|_{\mathbb{G}(T,\Omega)}\big)e^{C^{(1)}_{\mathrm{tra}}T^{1-\frac{1}{p}}\|\tilde w + u_*\|_{\mathbb{E}_u(T,\Omega)}}\\
&\quad\le C^{(1)}_{\mathrm{tra}}\big(\|\tau_0\|_{H^1_p(\Omega)} + O(T)\big)e^{O(T)},
\end{aligned}
$$

where the constant $C^{(1)}_{\mathrm{tra}}$ is given in the proposition on the transport equation, and

$$
\begin{aligned}
&\|\partial_t\tau\|_{T,\Omega,r,p}\\
&\quad= \|(\tilde\Phi_{0,\tau_0}(\tilde w, f_* + F_w(\tilde w) + F_\tau(\tilde\tau), G(\tilde w,\tilde\tau), h_* + H_w(\tilde w) + H_\tau(\tilde\tau)))_2\|_{\widehat{W}^1_r(0,T;L_p(\Omega))}\\
&\quad\le \|G(\tilde w,\tilde\tau)\|_{T,\Omega,r,p} + \|\tilde w + u_*\|_{T,\Omega,r,\infty}\|\tau\|_{L_\infty(0,T;H^1_p(\Omega))}\\
&\quad\le \|G(\tilde w,\tilde\tau)\|_{\mathbb{G}(T,\Omega)} + T^{\frac{1}{r}}\|\tilde w + u_*\|_{T,\Omega,\infty,\infty}\|\tau\|_{L_\infty(0,T;H^1_q(\Omega))}\\
&\quad\le \|G(\tilde w,\tilde\tau)\|_{\mathbb{G}(T,\Omega)} + T^{\frac{1}{r}}(C\|\tilde w\|_{{}_0\mathbb{E}_u(T,\Omega)} + \|u_*\|_{T_0,\Omega,\infty,\infty})R_2\\
&\quad\le O(T),
\end{aligned}
$$

where $O\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a function with $O(t) \to 0$ for $t \to 0$, which is independent of R_1, $0 < R_1 < R_0$. Setting $R_2 := 2C^{(1)}_{\mathrm{tra}}\|\tau_0\|_{H^1_p(\Omega)}$, $R_0, R_3, T_0 = 1$ and choosing first $R_1 > 0$ and then $T > 0$ sufficiently small, we find that $\Phi(\mathcal{K}(T, R_1, R_2, R_3))$ is contained in $\mathcal{K}(T, R_1, R_2, R_3)$.

Compactness of $\mathcal{K}(T, R_1, R_2, R_3)$ in $C([0,T], C^1(\overline{\Omega})) \times C([0,T], C(\overline{\Omega}))$

We show that $\mathcal{K}(T, R_1, R_2, R_3)$ with T, R_1, R_2, R_3 fixed as above, is compact in

$$\mathbb{Z}(T,\Omega) := \mathbb{Z}_u(T,\Omega) \times \mathbb{Z}_\tau(T,\Omega) := C([0,T], C^1(\overline{\Omega})) \times C([0,T], C(\overline{\Omega})).$$

Since the embeddings

$$\mathbb{E}_u(T,\Omega) \overset{c}{\hookrightarrow} \mathbb{Z}_u(T,\Omega) \quad \text{and} \quad \mathbb{E}_\tau(T,\Omega) \overset{c}{\hookrightarrow} \mathbb{Z}_\tau(T,\Omega)$$

are compact and $\mathcal{K}(T, R_1, R_2, R_3)$ is bounded in $\mathbb{E}_u(T,\Omega) \times \mathbb{E}_\tau(T,\Omega)$, $\mathcal{K}(T, R_1, R_2, R_3)$ is relatively compact in $\mathbb{Z}_u(T,\Omega) \times \mathbb{Z}_\tau(T,\Omega)$. It remains to show that $\mathcal{K}(T, R_1, R_2, R_3)$ is closed in $\mathbb{Z}(T,\Omega)$. Let $(w_\lambda, \tau_\lambda)_{\lambda\in\mathbb{N}} \subset \mathbb{E}_u(T,\Omega) \times \mathbb{E}_\tau(T,\Omega)$ be a sequence and $(w,\tau) \in \mathbb{Z}_u(T,\Omega) \times \mathbb{Z}_\tau(T,\Omega)$ with

$$\mathcal{K}(T, R_1, R_2, R_3) \ni (w_\lambda, \tau_\lambda) \to (w,\tau) \quad \text{in } \mathbb{Z}_u(T,\Omega) \times \mathbb{Z}_\tau(T,\Omega).$$

We show that $(w,\tau) \in \mathcal{K}(T, R_1, R_2, R_3)$. Since $\mathcal{K}(T, R_1, R_2, R_3) \subset \mathbb{E}_u(T,\Omega) \times \mathbb{E}_\tau(T,\Omega)$ is closed, bounded, and convex, there exists $(v,\sigma) \in \mathcal{K}(T, R_1, R_2, R_3)$ and a subsequence (which we again denote by $(w_\lambda, \tau_\lambda)_{\lambda\in\mathbb{N}}$), such that

$$w_\lambda \rightharpoonup v \text{ in } \mathbb{E}_u(T,\Omega), \quad \tau_\lambda \overset{*}{\rightharpoonup} \sigma \text{ in } \mathbb{E}_\tau(T,\Omega), \quad \text{and} \quad \tau_\lambda \rightharpoonup \sigma \text{ in } H^1_p(0,T;L_p(\Omega)) \cap L_p(0,T;H^1_p(\Omega)).$$

By the compactness of the embeddings

$$\mathbb{E}_u(T,\Omega) \overset{c}{\hookrightarrow} \mathbb{Z}_u(T,\Omega) \quad \text{and} \quad H^1_p(0,T;L_p(\Omega)) \cap L_p(0,T;H^1_p(\Omega)) \overset{c}{\hookrightarrow} \mathbb{Z}_\tau(T,\Omega),$$

the strong convergence

$$w_\lambda \to v \quad \text{in } \mathbb{Z}_u(T,\Omega) \quad \text{and} \quad \tau_\lambda \to \sigma \quad \text{in } \mathbb{Z}_\tau(T,\Omega)$$

follows. Since limits are unique, we can conclude that $(w,\tau) = (v,\sigma)$. Consequently $\mathcal{K}(T, R_1, R_2, R_3)$ is relatively compact and closed and therefore compact.

Continuity of $\Phi|_{\mathcal{K}(T,R_1,R_2,R_3)} \colon \mathcal{K}(T, R_1, R_2, R_3) \to \mathcal{K}(T, R_1, R_2, R_3)$ in topology of $\mathbb{Z}(T,\Omega)$

We show that $\Phi\colon \mathcal{K}(T, R_1, R_2, R_3) \to \mathcal{K}(T, R_1, R_2, R_3)$ is continuous in the topology of $\mathbb{Z}(T,\Omega)$. In particular, we show that Φ is sequentially continuous. Let $(\tilde{w}_\lambda, \tilde{\tau}_\lambda)_{\lambda\in\mathbb{N}} \subset \mathcal{K}(T, R_1, R_2, R_3)$ be a sequence converging to $(\tilde{w}, \tilde{\tau}) \in \mathcal{K}(T, R_1, R_2, R_3)$ in $\mathbb{Z}(T,\Omega)$, i.e.

$$\mathcal{K}(T, R_1, R_2, R_3) \ni (\tilde{w}_\lambda, \tilde{\tau}_\lambda) \to (\tilde{w}, \tilde{\tau}) \quad \text{in } \mathbb{Z}(T,\Omega),$$

and let

$$(w_\lambda, \tau_\lambda) := \Phi(\tilde{w}_\lambda, \tilde{\tau}_\lambda), \quad \lambda \in \mathbb{N} \quad \text{and} \quad (w,\tau) := \Phi(\tilde{w}, \tilde{\tau})$$

be the solution of the linearized problem. We show that $(w_\lambda, \tau_\lambda) \to (w,\tau)$ in $\mathbb{Z}(T,\Omega)$. Since $\mathcal{K}(T, R_1, R_2, R_3)$ is closed, bounded, and convex, there exists a subsequence of $(w_\lambda, \tau_\lambda)_{\lambda\in\mathbb{N}}$ and a subsequence of $(\tilde{w}_\lambda, \tilde{\tau}_\lambda)_{\lambda\in\mathbb{N}}$ (which we again denote by $(w_\lambda, \tau_\lambda)_{\lambda\in\mathbb{N}}$ and $(\tilde{w}_\lambda, \tilde{\tau}_\lambda)_{\lambda\in\mathbb{N}}$) and functions $(v,\sigma), (\tilde{v}, \tilde{\sigma}) \in \mathcal{K}(T, R_1, R_2, R_3)$, such that

$$w_\lambda \rightharpoonup v \text{ in } \mathbb{E}_u(T,\Omega), \quad \tau_\lambda \overset{*}{\rightharpoonup} \sigma \text{ in } \mathbb{E}_\tau(T,\Omega), \text{ and } \tau_\lambda \rightharpoonup \sigma \text{ in } H^1_p(0,T;L_p(\Omega)) \cap L_p(0,T;H^1_p(\Omega)),$$

as well as

$$\tilde{w}_\lambda \rightharpoonup \tilde{v} \text{ in } \mathbb{E}_u(T,\Omega), \quad \tilde{\tau}_\lambda \overset{*}{\rightharpoonup} \tilde{\sigma} \text{ in } \mathbb{E}_\tau(T,\Omega), \text{ and } \tilde{\tau}_\lambda \rightharpoonup \tilde{\sigma} \quad \text{in } H^1_p(0,T;L_p(\Omega)) \cap L_p(0,T;H^1_p(\Omega)).$$

By the compactness of the embeddings

$$\mathbb{E}_u(T,\Omega) \overset{c}{\hookrightarrow} \mathbb{Z}_u(T,\Omega) \quad \text{and} \quad H^1_p(0,T;L_p(\Omega)) \cap L_p(0,T;H^1_p(\Omega)) \overset{c}{\hookrightarrow} \mathbb{Z}_\tau(T,\Omega),$$

we deduce the strong convergences

$$w_\lambda \to v \text{ in } \mathbb{Z}_w(T,\Omega), \quad \tau_\lambda \to \sigma \text{ in } \mathbb{Z}_\tau(T,\Omega), \quad \tilde{w}_\lambda \to \tilde{v} \text{ in } \mathbb{Z}_w(T,\Omega), \text{ and } \tilde{\tau}_\lambda \to \tilde{\sigma} \quad \text{in } \mathbb{Z}_\tau(T,\Omega).$$

Therefore $(\tilde{w}, \tilde{\tau}) = (\tilde{v}, \tilde{\sigma})$ by the uniqueness of limits. It remains to show that $(w,\tau) = (v,\sigma)$. We will show this equality by proving the (w,τ) and (v,σ) solve the same uniquely solvable system of partial differential equations, i.e. we prove that

$$(w,\tau) = (v,\sigma) = \Phi(\tilde{w}, \tilde{\tau}) = \tilde{\Phi}_{0,\tau_0}(\tilde{w}, f_* + F_w(\tilde{w}) + F_\tau(\tilde{\tau}), G(\tilde{w}, \tilde{\tau}), h_* + H_w(\tilde{w} + H_\tau(\tilde{\tau})).$$

By definition, $(w_\lambda, \tau_\lambda)$ is the solution of

$$
(2.9) \qquad \begin{cases}
\rho\partial_t w_\lambda + \mathcal{A}(Eu_*)w_\lambda + \nabla\psi_\lambda &= f_* + F_w(\tilde{w}_\lambda) + F_\tau(\tilde{\tau}_\lambda) & \text{in } (0,T)\times\Omega,\\
\operatorname{div} w_\lambda &= 0 & \text{in } (0,T)\times\Omega,\\
\partial_t\tau_\lambda + (\tilde{w}_\lambda + u_*).\nabla\tau_\lambda &= G(\tilde{w}_\lambda, \tilde{\tau}_\lambda) & \text{in } (0,T)\times\Omega,\\
w_\lambda &= 0 & \text{on } (0,T)\times\Gamma_D,\\
(w_\lambda\cdot\nu, \mathcal{B}_{\mathcal{S}}(Eu_*)w_\lambda) &= (0, h_* + H_w(\tilde{w}_\lambda) + H_\tau(\tilde{\tau}_\lambda)) & \text{on } (0,T)\times\Gamma_S,\\
w_\lambda(0) &= 0 & \text{in } \Omega,\\
\tau_\lambda(0) &= \tau_0 & \text{in } \Omega.
\end{cases}
$$

We investigate the limit $\lambda \to \infty$ in each term of the equation. We have, due to the strong convergence of $(w_\lambda, \tau_\lambda)_{\lambda\in\mathbb{N}} \to (v, \sigma)$ in $\mathbb{Z}(T, \Omega)$, the equality

$$(0, \tau_0) = \lim_{\lambda\to\infty}(w_\lambda(0), \tau_\lambda(0)) = (v(0), \sigma(0)) \quad \text{in } C(\overline{\Omega}),$$

the convergence of the divergence free condition

$$0 = \lim_{\lambda\to\infty} \operatorname{div} w_\lambda = \operatorname{div} v \quad \text{in } C([0,T], C(\overline{\Omega})),$$

the convergence of the boundary conditions on $(0,T)\times\Gamma_D$,

$$0 = \lim_{\lambda\to\infty} w_\lambda = v \quad \text{on } C([0,T], C(\Gamma_D)),$$

and the convergence of the boundary condition on $(0,T)\times\Gamma_S$

$$0 = \lim_{\lambda\to\infty} w_\lambda\cdot\nu = v\cdot\nu \quad \text{on } C([0,T], C(\Gamma_S)).$$

Moreover, since the boundary operator $\mathcal{B}_{\mathcal{S}}(Eu_*)$ is of first order and the coefficients are continuous, we obtain on $(0,T)\times\Gamma_S$ that

$$\mathcal{B}_{\mathcal{S}}(Eu_*)w_\lambda \to \mathcal{B}_{\mathcal{S}}(Eu_*)v \quad \text{on } C([0,T], C(\Gamma_S)).$$

The next subject is the convergence of the terms $H_w(\tilde{w}_\lambda)$ and $H_\tau(\tilde{\tau}_\lambda)$. By continuity of α and μ as well as strong convergence of $\tilde{w}_\lambda \to \tilde{w}$ in $\mathbb{Z}_u(T,\Omega)$ and $\tilde{\tau}_\lambda \to \tilde{\tau}$ in $\mathbb{Z}_\tau(T,\Omega)$, it follows that

$$\lim_{\lambda\to\infty} H_w(\tilde{w}_\lambda) = H_w(\tilde{w}) \quad \text{and} \quad \lim_{\lambda\to\infty} H_\tau(\tilde{\tau}_\lambda) = H_w(\tilde{\tau}).$$

Thus, we proved that v and w fulfill also the same boundary condition

$$(w\cdot\nu, \mathcal{B}_{\mathcal{S}}(Eu_*)w) = (0, h_* + H_w(\tilde{w}) + H_\tau(\tilde{\tau})) = (v\cdot\nu, \mathcal{B}_{\mathcal{S}}(Eu_*)v) \quad \text{on } (0,T)\times\Gamma_S.$$

To analyse the limit in the generalized Stokes equation (the first equation of (2.9)), we test this equation with a smooth, divergence free function $\varphi_1 \in C_c^\infty((0,T); C_{c,\sigma}^\infty(\Omega))$, and obtain

$$(\partial_t w_\lambda|\varphi_1)_{T,\Omega} + (\mathcal{A}(Eu_*)w_\lambda|\varphi_1)_{T,\Omega} = (f_*|\varphi_1)_{T,\Omega} + (F_w(\tilde{w}_\lambda)|\varphi_1)_{T,\Omega} + (F_\tau(\tilde{\tau}_\lambda)|\varphi_1)_{T,\Omega}.$$

We pass to the limit in each summand separately. Due to the weak convergence $\partial_t w_\lambda \rightharpoonup \partial_t v$ in $L_p(0,T;L_p(\Omega))$, we obtain

$$\lim_{\lambda\to\infty}(\partial_t w_\lambda|\varphi_1)_{T,\Omega} = (\partial_t v|\varphi_1)_{T,\Omega}.$$

Using the weak convergence $\nabla^2 w_\lambda \rightharpoonup \nabla^2 v$ in $L_p(0,T;L_p(\Omega))$ and the continuity of the coefficients $\mathcal{A}^{l,m}_{j,k}(Eu_*)$ (and thus $\mathcal{A}^{l,m}_{j,k}(Eu_*)\varphi_{1,j} \in L_{p'}(0,T;L_{p'}(\Omega))$ for $1 < p' < \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$, since $(0,T) \times \Omega$ is bounded) of the second order differential operator $\mathcal{A}(Eu_*)$, it follows that

$$\begin{aligned}
\lim_{\lambda\to\infty} \big(\mathcal{A}(Eu_*)w_\lambda\big|\varphi_1\big)_{T,\Omega} &= \lim_{\lambda\to\infty} \sum_{j,k,l,m=1}^{n} -\big(\mathcal{A}^{l,m}_{j,k}(Eu_*)\partial_l\partial_m w_{\lambda,k}\big|\varphi_{1,j}\big)_{T,\Omega} \\
&= -\lim_{\lambda\to\infty} \sum_{j,k,l,m=1}^{n} \big(\partial_l\partial_m w_{\lambda,k}\big|\mathcal{A}^{l,m}_{j,k}(Eu_*)\varphi_{1,j}\big)_{T,\Omega} \\
&= -\sum_{j,k,l,m=1}^{n} \big(\partial_l\partial_m v_k\big|\mathcal{A}^{l,m}_{j,k}(Eu_*)\varphi_{1,j}\big)_{T,\Omega} \\
&= \big(\mathcal{A}(Eu_*)v\big|\varphi_1\big)_{T,\Omega}.
\end{aligned}$$

We recall the definition

$$F_w(\tilde{w}_\lambda) = (\mathcal{A}(Eu_*) - \mathcal{A}(E(\tilde{w}_\lambda + u_*)))(\tilde{w}_\lambda + u_*) - \rho u_*.\nabla\tilde{w}_\lambda - \rho\tilde{w}_\lambda.\nabla u_* - \rho\tilde{w}_\lambda.\nabla\tilde{w}_\lambda.$$

Each term is the product of sequences $\tilde{f}_\lambda \to \tilde{f}$ in $C([0,T],C(\overline{\Omega}))$ (since $\tilde{f} \in C([0,T],C(\overline{\Omega}))$, we have $\tilde{f}\varphi_1 \in L_{p'}(0,T;L_{p'}(\Omega))$ for $1 < p' < \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$) and $\tilde{g}_\lambda \rightharpoonup \tilde{g}$ in $L_p(0,T;L_p(\Omega))$, which is additionally uniformly bounded, i.e. $\|g_\lambda\|_{T,\Omega,p,p} < R_1$ for $\lambda \in \mathbb{N}$. In this situation, it holds that

$$\begin{aligned}
\lim_{\lambda\to\infty} \big(\tilde{f}_\lambda\tilde{g}_\lambda\big|\varphi_1\big)_{T,\Omega} - \big(\tilde{f}\tilde{g}\big|\varphi_1\big)_{T,\Omega} &= \lim_{\lambda\to\infty} \big((\tilde{f}_\lambda - \tilde{f})\tilde{g}_\lambda\big|\varphi_1\big)_{T,\Omega} + \lim_{\lambda\to\infty} \big(\tilde{f}(\tilde{g}_\lambda - \tilde{g})\big|\varphi_1\big)_{T,\Omega} \\
&= \lim_{\lambda\to\infty} \big((\tilde{f}_\lambda - \tilde{f})\big|\tilde{g}_\lambda\varphi_1\big)_{T,\Omega} + \lim_{\lambda\to\infty} \big((\tilde{g}_\lambda - \tilde{g})\big|\tilde{f}\varphi_1\big)_{T,\Omega} \\
&= 0.
\end{aligned} \tag{2.10}$$

This shows

$$\lim_{\lambda\to\infty} \big(F_w(\tilde{w}_\lambda)\big|\varphi_1\big) = \big(F_w(\tilde{w})\big|\varphi_1\big).$$

Next, we investigate the term $(F_\tau(\tilde{\tau}_\lambda)|\varphi_1)_{T,\Omega}$. Using integration by parts and the strong convergence $\tilde{\tau}_\lambda \to \tilde{\tau}$ in $\mathbb{Z}_\tau(T,\Omega)$ gives

$$\begin{aligned}
\lim_{\lambda\to\infty} \big(F_\tau(\tilde{\tau}_\lambda)\big|\varphi_1\big)_{T,\Omega} &= \lim_{\lambda\to\infty} \big(\operatorname{Div}(\mu(\tilde{\tau}_\lambda) - \mu(\tau_0))\big|\varphi_1\big)_{T,\Omega} \\
&= -\lim_{\lambda\to\infty} \big(\mu(\tilde{\tau}_\lambda) - \mu(\tau_0) : \nabla\varphi_1\big)_{T,\Omega} \\
&= -\big(\mu(\tilde{\tau}) - \mu(\tau_0) : \nabla\varphi_1\big)_{T,\Omega} \\
&= \big(\operatorname{Div}(\mu(\tilde{\tau}) - \mu(\tau_0))\big|\varphi_1\big)_{T,\Omega} \\
&= \big(F_\tau(\tilde{\tau})\big|\varphi_1\big)_{T,\Omega}.
\end{aligned}$$

In summary, we proved

$$(\partial_t v|\varphi_1)_{T,\Omega} + (\mathcal{A}(Eu_*)v|\varphi_1)_{T,\Omega} = (f_*|\varphi_1)_{T,\Omega} + (F_w(\tilde{w})|\varphi_1)_{T,\Omega} + (F_\tau(\tilde{\tau})|\varphi_1)_{T,\Omega}, \qquad \varphi_1 \in C_c^\infty((0,T);C_{c,\sigma}^\infty(\Omega)),$$

and thus, there exists a gradient field $\nabla\tilde{\psi} \in L_p(0,T;L_p(\Omega))$ with

$$\partial_t v + \mathcal{A}(Eu_*)v + \nabla\tilde{\psi} = f_* + F_w(\tilde{w}) + F_\tau(\tilde{\tau}).$$

Next, we test the transport equation in (2.9) with $\varphi_2 \in C_c^\infty((0,T);C_c^\infty(\Omega))$ and obtain

$$\big(\partial_t \tau_\lambda : \varphi_2\big)_{T,\Omega} + \big((\tilde{w}_\lambda + u_*).\nabla\tau_\lambda : \varphi_2\big)_{T,\Omega} = \big(G(\tilde{w}_\lambda, \tilde{\tau}_\lambda) : \varphi_2\big)_{T,\Omega}.$$

By the convergence $\partial_t\tau_\lambda \overset{*}{\rightharpoonup} \partial_t\sigma$, we have

$$\lim_{\lambda\to\infty} \big(\partial_t\tau_\lambda : \varphi_2\big)_{T,\Omega} = \big(\partial_t\sigma : \varphi_2\big)_{T,\Omega}.$$

Using the strong convergence $\tilde{w}_\lambda \to \tilde{w}$ in $C([0,T],C^1(\overline{\Omega}))$, the weak convergence $\nabla\tau_\lambda \rightharpoonup \nabla\sigma$ in $L_p(0,T;L_p(\Omega))$ and (2.10), it follows that

$$\lim_{\lambda\to\infty} \big((\tilde{w}_\lambda + u_*).\nabla\tau_\lambda : \varphi_2\big)_{T,\Omega} = \big((\tilde{w} + u_*).\nabla\sigma : \varphi_2\big)_{T,\Omega}.$$

By the strong convergences $\tilde{w}_\lambda \to \tilde{w}$ in $C([0,T],C^1(\overline{\Omega}))$ and $\tilde{\tau}_\lambda \to \tilde{\tau}$ in $C([0,T],C(\overline{\Omega}))$, we deduce that

$$\lim_{\lambda\to\infty} \big(G(\tilde{w}_\lambda, \tilde{\tau}_\lambda) : \varphi_2\big)_{T,\Omega} = \big(G(\tilde{w}, \tilde{\tau}) : \varphi_2\big)_{T,\Omega}.$$

This shows that σ satisfies

$$\partial_t\sigma + (\tilde{w} + u_*).\nabla\sigma = G(\tilde{w}, \tilde{\tau}).$$

In summary, we proved $(w,\tau) = \Phi(\tilde{w},\tilde{\tau}) = (v,\sigma)$, and hence $(w,\tau) = (v,\sigma)$.

Application of the fixed point argument and completion of the existence result

Since $(0,\tau_0) \in \mathcal{K}(T,R_1,R_2,R_3)$, $\mathcal{K}(T,R_1,R_2,R_3)$ is not empty, Schauder's fixed point theorem (Proposition 1.11) guarantees the existence of a fixed point of Φ and equivalently, the existence of a solution

$$(u,\tau) \in H_p^1(0,T;L_p(\Omega)) \cap L_p(0,T;H_p^2(\Omega)) \times \widehat{H}_r^1(0,T;L_p(\Omega)) \cap L_\infty(0,T;L_p(\Omega))$$

of (2.1). The corresponding pressure term is defined by

$$\nabla\pi = \big(I - P_p\big)\big(\operatorname{Div}\mu(\tau) - \rho(\partial_t u + u.\nabla u) + \operatorname{Div} 2\alpha(|Eu|^2)Eu\big) \in L_p(0,T;L_p(\Omega)).$$

It remains to prove the additional regularity of the elastic part of the stress τ. The function τ in particular fulfills the transport equation

$$\partial_t\tau + (w + u_*).\nabla\tau = G(w,\tau).$$

In Lemma 2.3, we proved $G(w,\tau) \in L_\infty(0,T;L_p(\Omega))$. By the proposition on the transport equation (Proposition 1.10), we can conclude that $\partial_t\tau \in L_\infty(0,T;L_p(\Omega))$.

Uniqueness with an energy method

We already proved the existence of a solution of (2.1). It remains to show the uniqueness of this solution. According to the boundedness of $(0,T)$ and the domain Ω, it is sufficient to guarantee the uniqueness of a solution in the L_2-setting. In the end, it is found more convenient to consider the original problem (2.1) instead of the reduced, equivalent problem (2.6).

Let

$$(u_j,\pi_j,\tau_j)\in\mathbb{E}_u(T,\Omega)\times L_p(0,T;\widehat{H}^1_p(\Omega))\times\mathbb{E}_\tau(T,\Omega),\quad j=1,2, \tag{2.11}$$

be two solutions of (2.1), i.e.

$$\begin{cases}\rho(\partial_t u_j+u_j.\nabla u_j)-\operatorname{Div}2\alpha(|Eu_j|^2)Eu_j+\nabla\pi_j &= \operatorname{Div}\mu(\tau_j)+f & \text{in } (0,T)\times\Omega,\\ \operatorname{div}u_j &= 0 & \text{in } (0,T)\times\Omega,\\ \partial_t\tau_j+u_j.\nabla\tau_j &= g(\nabla u_j,\tau_j) & \text{in } (0,T)\times\Omega,\\ u_j &= 0 & \text{on } (0,T)\times\Gamma_D,\\ (u_j\cdot\nu,[2\alpha(|Eu_j|^2)Eu_j\nu+\mu(\tau_j)\nu]_{\tan}) &= 0 & \text{on } (0,T)\times\Gamma_S,\\ u_j(0) &= u_0 & \text{in } \Omega,\\ \tau_j(0) &= \tau_0 & \text{in } \Omega.\end{cases}$$

We denote the difference of two solutions by

$$(u_{12},\pi_{12},\tau_{12}):=(u_2-u_1,\pi_2-\pi_1,\tau_2-\tau_1).$$

Our aim is to show that $(u_{12},\tau_{12})=(0,0)$. Since the time interval $(0,T)$ and Ω are bounded, we conclude by Hölder's inequality that

$$\begin{aligned}&u_j\in H^1_2(0,T;L_2(\Omega))\cap L_2(0,T;H^2_2(\Omega)),\quad \pi_j\in L_2(0,T;\widehat{H}^1_2(\Omega))\\ &\text{and}\quad \tau_j\in H^1_2(0,T;L_2(\Omega))\cap L_2(0,T;H^1_2(\Omega)),\quad j=1,2.\end{aligned}$$

The difference of the velocity fields u_{12} fulfills the generalized Stokes problem

$$\begin{aligned}\rho(\partial_t u_{12}+u_1.\nabla u_{12})-\operatorname{Div}\big(2\alpha(|Eu_2|^2)Eu_2-2\alpha(|Eu_1|^2)Eu_1+\mu(\tau_2)-\mu(\tau_1)\big)+\nabla\pi_{12}&\\ =-\rho u_{12}.\nabla u_2\quad \text{in } (0,T)\times\Omega.&\end{aligned}$$

At a fixed time $0<t<T$, we multiply this equation with $u_{12}(t)$ and integrate over Ω. The advection term on the left-hand side disappears due to $u_1\cdot\nu=0$ on $\partial\Omega$ and $\operatorname{div}u_1=0$ in Ω, since

$$\begin{aligned}\int_\Omega(u_1(t).\nabla u_{12}(t))u_{12}(t)dx&=\int_\Omega\sum_{j,k=1}^n u_{1,j}(t)(\partial_j u_{12,k}(t))u_{12,k}(t)dx=\frac{1}{2}\int_\Omega\sum_{j=1}^n u_{1,j}(t)\partial_j|u_{12}(t)|^2dx\\ &=-\frac{1}{2}\int_\Omega\operatorname{div}u_1(t)|u_{12}(t)|^2+\int_{\partial\Omega}u_1(t)\cdot\nu|u_{12}(t)|^2dx=0,\end{aligned} \tag{2.12}$$

due to integration by parts as well as the pressure difference vanishes as a gradient field. We conclude that

$$\begin{aligned}&\frac{\rho}{2}\frac{d}{dt}\|u(t)\|^2_{\Omega,2}\\ &\quad-\int_\Omega\operatorname{Div}\big(2\alpha(|Eu_2(t)|^2)Eu_2(t)-2\alpha(|Eu_1(t)|^2)Eu_1(t)+\mu(\tau_2(t))-\mu(\tau_1(t))\big)\cdot u_{12}(t)dx\\ &=-\int_\Omega(u_{12}(t).\nabla u_2(t))\cdot u_{12}(t).\end{aligned} \tag{2.13}$$

To treat the integral term on the left-hand side, we integrate by parts and use $u_2 - u_1 = u_{12} = 0$ on Γ_D. The result is

$$
\begin{aligned}
&-\int_\Omega \operatorname{Div}\big(2\alpha(|Eu_2(t)|^2)Eu_2(t) - 2\alpha(|Eu_1(t)|^2)Eu_1(t) + \mu(\tau_2(t)) - \mu(\tau_1(t))\big)\cdot u_{12}(t)dx \\
&\quad = \int_\Omega \Big(\big(2\alpha(|Eu_2(t)|^2)Eu_2(t) - 2\alpha(|Eu_1(t)|^2)Eu_1(t) + \mu(\tau_2(t)) - \mu(\tau_1(t))\big) : \nabla u_{12}(t)\Big)dx \\
&\qquad + \int_{\partial\Omega} \big(2\alpha(|Eu_2(t)|^2)Eu_2(t) - 2\alpha(|Eu_1(t)|^2)Eu_1(t) + \mu(\tau_2(t)) - \mu(\tau_1(t))\big)\nu\cdot u_{12}(t)dx \\
&\quad = \int_\Omega \Big(\big(2\alpha(|Eu_2(t)|^2)Eu_2(t) - 2\alpha(|Eu_1(t)|^2)Eu_1(t) + \mu(\tau_2(t)) - \mu(\tau_1(t))\big) : \nabla u_{12}(t)\Big)dx \\
&\qquad + \int_{\Gamma_S} \big(2\alpha(|Eu_2(t)|^2)Eu_2(t) - 2\alpha(|Eu_1(t)|^2)Eu_1(t) + \mu(\tau_2(t)) - \mu(\tau_1(t))\big)\nu\cdot u_{12}(t)dx.
\end{aligned}
$$

Next, we show that the remaining boundary integral also vanishes. According to the boundary condition on Γ_S

$$
\big(u_{12}\cdot\nu, \big[2\alpha(|Eu_2|^2)Eu_2\nu - 2\alpha(|Eu_1|^2)Eu_1\nu + \mu(\tau_2)\nu - \mu(\tau_1)\nu\big]_{\tan}\big) = 0 \quad \text{on } \Gamma_S,
$$

we compute

$$
\begin{aligned}
&\int_{\Gamma_S} \big(2\alpha(|Eu_2(t)|^2)Eu_2(t) - 2\alpha(|Eu_1(t)|^2)Eu_1(t) + \mu(\tau_2(t)) - \mu(\tau_1(t))\big)\nu\cdot u_{12}(t)dx \\
&\quad = \int_{\Gamma_S} \big[\big(2\alpha(|Eu_2(t)|^2)Eu_2(t) - 2\alpha(|Eu_1(t)|^2)Eu_1(t) + \mu(\tau_2(t)) - \mu(\tau_1(t))\big)\nu\big]_{\tan}\cdot\big[u_{12}(t)\big]_{\tan}dx \\
&\qquad + \int_{\Gamma_S} \big[\big(2\alpha(|Eu_2(t)|^2)Eu_2(t) - 2\alpha(|Eu_1(t)|^2)Eu_1(t) + \mu(\tau_2(t)) - \mu(\tau_1(t))\big)\nu\big]_{\nu}\cdot\big[u_{12}(t)\big]_{\nu}dx \\
&\quad = 0.
\end{aligned}
$$

By the symmetry $2\alpha(|Eu_j(t)|^2)Eu_j(t) = 2\alpha(|Eu_j(t)|^2)Eu_j(t)^T$, $j = 1, 2$, it follows that

$$
\begin{aligned}
&-\int_\Omega \operatorname{Div}\big(2\alpha(|Eu_2(t)|^2)Eu_2(t) - 2\alpha(|Eu_1(t)|^2)Eu_1(t) + \mu(\tau_2(t)) - \mu(\tau_1(t))\big)\cdot u_{12}(t)dx \\
&\quad = \int_\Omega \big(\big(2\alpha(|Eu_2(t)|^2)Eu_2(t) - 2\alpha(|Eu_1(t)|^2)Eu_1(t)\big) : Eu_{12}(t)\big)dx \\
&\qquad + \int_\Omega \big((\mu(\tau_2(t)) - \mu(\tau_1(t))) : \nabla u_{12}(t)\big)dx.
\end{aligned} \tag{2.14}
$$

The next lemma is used, to investigate the first integral on the right-hand side (2.14). Basically, we prove that the strong monotonicity of $Eu \mapsto \alpha(|Eu|^2)Eu$ is implied by the structure condition (2.4).

Lemma 2.5. *Let $n \in \mathbb{N}$, $n \geq 2$, $C_* > 0$ and let $\alpha \in C^1([0,\infty))$ satisfy the structure condition*

$$
\alpha(s) > 0 \quad \text{and} \quad \alpha(s) + 2s\alpha'(s) > 0, \qquad s \geq 0.
$$

Then, there exists a constant $\alpha_0 > 0$, such that for all $X_1, X_2 \in \mathbb{R}^{n\times n}$ with $|X_j| \leq C_*$, $j = 1, 2$, the strong monotonicity estimate

$$
\big((2\alpha(|X_2|^2)X_2 - 2\alpha(|X_1|^2)X_1) : (X_2 - X_1)\big) \geq 2\alpha_0\big|X_2 - X_1\big|^2.
$$

holds.

Proof. Without restriction of generality, we consider α on the finite interval $[0, C_*^2]$. There exists a constant $\alpha_0 > 0$ with

$$\alpha(s) > \alpha_0 \quad \text{and} \quad \alpha(s) + 2s\alpha'(s) > \alpha_0, \qquad 0 \le s \le C_*^2.$$

We define $\tilde{\alpha} = \alpha - \alpha_0$ and conclude that

$$\tilde{\alpha}(s) > 0 \quad \text{and} \quad \tilde{\alpha}(s) + 2s\tilde{\alpha}'(s) > 0, \qquad 0 \le s \le C_*^2.$$

Hence, we have

$$\begin{aligned}
&\big((2\alpha(|X_2|^2)X_2 - 2\alpha(|X_1|^2)X_1) : (X_2 - X_1)\big) \\
&\quad = \big((2\alpha_0(X_2 - X_1) + 2\tilde{\alpha}(|X_2|^2)X_2 - 2\tilde{\alpha}(|X_1|^2)X_1) : (X_2 - X_1)\big) \\
&\quad = 2\alpha_0\big|X_2 - X_1\big|^2 + \big((2\tilde{\alpha}(|X_2|^2)X_2 - 2\tilde{\alpha}(|X_1|^2)X_1) : (X_2 - X_1)\big).
\end{aligned}$$

It remains to prove that the second summand is not negative:

$$\begin{aligned}
&\big((\tilde{\alpha}(|X_2|^2)X_2 - \tilde{\alpha}(|X_1|^2)X_1) : (X_2 - X_1)\big) \\
&\quad = \tilde{\alpha}(|X_1|^2)|X_1|^2 + \tilde{\alpha}(|X_2|^2)|X_2|^2 - (\tilde{\alpha}(|X_1|^2) + \tilde{\alpha}(|X_2|^2))(X_1 : X_2) \\
&\quad \ge \tilde{\alpha}(|X_1|^2)|X_1|^2 + \tilde{\alpha}(|X_2|^2)|X_2|^2 - (\tilde{\alpha}(|X_1|^2) + \tilde{\alpha}(|X_2|^2))|X_1||X_2| \\
&\quad = \big(\tilde{\alpha}(|X_2|^2)|X_2| - \tilde{\alpha}(|X_1|^2)|X_1|\big)\big(|X_2| - |X_1|\big) \\
&\quad = \Big(\int_{|X_1|}^{|X_2|} \frac{d}{ds'}\tilde{\alpha}(s'^2)s'ds'\Big)\big(|X_2| - |X_1|\big) \\
&\quad = \Big(\int_{|X_1|}^{|X_2|} \tilde{\alpha}(s'^2) + 2\tilde{\alpha}'(s'^2)s'^2 ds'\Big)\big(|X_2| - |X_1|\big) \\
&\quad \ge 0.
\end{aligned}$$

□

In (2.11), we fixed two solutions. This solutions belong to the space $u_j \in \mathbb{E}_u(T, \Omega)$, $j = 1, 2$, and thus, we can define the constant

$$C_* := \max_{j=1,2} \big\{\|u_j\|_{L_\infty(0,T;W^1_\infty(\Omega))}, \|\tau_j\|_{L_\infty(0,T;L_\infty(\Omega))}\big\},$$

by the proposition on embedding theorems (Proposition 1.14). Combining (2.13), (2.14), and Lemma 2.5, it follows by the mean value theorem that

$$\begin{aligned}
&\frac{\rho}{2}\frac{d}{dt}\|u_{12}(t)\|_{\Omega,2}^2 + 2\alpha_0\|Eu_{12}(t)\|_{\Omega,2}^2 \\
&\quad \le \int_\Omega |((\mu(\tau_2(t)) - \mu(\tau_1(t))) : \nabla u_{12})|dx - \int_\Omega |(u_{12}(t).\nabla u_2(t)) \cdot u_{12}(t))|dx \\
&\quad \le \|\mu(\tau_2(t)) - \mu(\tau_1(t))\|_{\Omega,2}\|\nabla u_{12}(t)\|_{\Omega,2} + \|u_2\|_{L_\infty(0,T;W^1_\infty(\Omega))}\|u_{12}(t)\|_{\Omega,2}^2 \\
&\quad \le \sup_{|\eta|<C_*} |(\nabla\mu)(\eta)|\|\tau_{12}(t)\|_{\Omega,2}\|\nabla u_{12}(t)\|_{\Omega,2} + C_*\|u_{12}(t)\|_{\Omega,2}^2 \\
&\quad \le C(\|\tau_{12}(t)\|_{\Omega,2}\|\nabla u_{12}(t)\|_{\Omega,2} + \|u_{12}(t)\|_{\Omega,2}^2), \quad 0 < t < T.
\end{aligned} \tag{2.15}$$

Now, the task is to replace the term $\|Eu_{12}(t)\|_{\Omega,2}$ by $\|u_{12}\|_{H^1_2(\Omega)}$. Unfortunately, due to the boundary condition on Γ_S, we cannot apply Korn's first inequality. Therefore, it is necessary to include the L_2-norm of u in the estimate on the left-hand side and use Korn's second inequality (see Proposition 1.20). Integrating (2.15) in time, we obtain

$$\frac{\rho}{2}\|u_{12}(t')\|^2_{\Omega,2} + 2\alpha_0\|Eu_{12}\|^2_{t',\Omega,2,2} \le C(\|\tau_{12}\|_{t,\Omega,2,2}\|\nabla u_{12}\|_{t,\Omega,2,2} + \|u_{12}\|^2_{t,\Omega,2,2}), \quad 0 < t' \le t < T.$$

In a first step, we take the supremum over $0 < t' < t$ in the previous equation and in a second step, we choose $t' = t$. Adding the results gives

$$\frac{\rho}{2}\|u_{12}(t)\|^2_{\Omega,2} + 2\alpha_0\|Eu_{12}\|^2_{t,\Omega,2,2} + \frac{\rho}{2}\|u_{12}\|^2_{t,\Omega,\infty,2} \le C(\|\tau_{12}\|_{t,\Omega,2,2}\|\nabla u_{12}\|_{t,\Omega,2,2} + \|u_{12}\|^2_{t,\Omega,2,2}), \quad 0 < t < T.$$

By Korn's second inequality (see Proposition 1.20) as well as Young's inequality, we deduce that

$$\begin{aligned}\|u_{12}(t)\|^2_{\Omega,2} + \|u_{12}\|^2_{L_2(0,t;H^1_2(\Omega))} &\le C(\|\tau_{12}\|_{t,\Omega,2,2}\|\nabla u_{12}\|_{t,\Omega,2,2} + \|u_{12}\|^2_{t,\Omega,2,2}) \\ &\le \frac{1}{2}\|\nabla u_{12}\|^2_{t,\Omega,2,2} + C(\|\tau_{12}\|^2_{t,\Omega,2,2} + \|u_{12}\|^2_{t,\Omega,2,2}).\end{aligned}$$

Absorbing the gradient term yields

$$\|u_{12}(t)\|^2_{\Omega,2} + \frac{1}{2}\|\nabla u_{12}\|^2_{t,\Omega,2,2} \le C\int_0^t \|\tau_{12}(t')\|^2_{\Omega,2} + \|u_{12}(t')\|^2_{\Omega,2})dt'. \tag{2.16}$$

Next, we investigate $\tau_{12} = \tau_2 - \tau_1$. This difference fulfills the transport equation

$$\partial_t\tau_{12} + u_1.\nabla\tau_{12} = g(\nabla u_2, \tau_2) - g(\nabla u_1, \tau_1) - u_{12}.\nabla\tau_2 \quad \text{in } (0,T)\times\Omega. \tag{2.17}$$

We fix a time $0 < t' < T$ and test this equation with $\tau(t')$ to obtain an a-priori estimate. Since $\operatorname{div} u_1 = 0$ in Ω and $u_1\cdot\nu = 0$ on $\partial\Omega$, the advection term vanishes (similar to (2.12)), and it remains

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\tau_{12}(t')\|_2^2 \\ &\quad= \int_\Omega \big((g(\nabla u_2(t'),\tau_2(t')) - g(\nabla u_1(t'),\tau_1(t'))) : \tau_{12}(t')\big)dx - \int_\Omega \big(u_{12}(t').\nabla\tau_2(t') : \tau_{12}(t')\big)dx \\ &\quad\le \|g(\nabla u_2(t'),\tau_2(t')) - g(\nabla u_1(t'),\tau_1(t'))\|_{\Omega,2}\|\tau_{12}(t')\|_{\Omega,2} + \|u_{12}(t')\|_{\Omega,q}\|\nabla\tau_2(t')\|_{\Omega,p}\|\tau_{12}(t')\|_{\Omega,2},\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$. By the mean value theorem, we obtain

$$\begin{aligned}&\|g(\nabla u_2(t'),\tau_2(t')) - g(\nabla u_1(t'),\tau_1(t'))\|_{\Omega,2} \\ &\qquad\le \sup_{|\eta_1|,|\eta_2|<C_*}|\nabla g(\eta_1,\eta_2)|(\|\nabla(u_2(t') - u_1(t'))\|_{\Omega,2} + \|\tau_2(t') - \tau_1(t')\|_{\Omega,2}).\end{aligned}$$

By Sobolev's embedding theorem, it follows that $H^1_2(\Omega) \to L_q(\Omega)$ (since $n < n+2 < p$). Therefore, by Youngs inequality, we deduce that

$$\begin{aligned}\frac{d}{dt}\|\tau_{12}(t')\|^2_{\Omega,2} &\le C(\|\nabla u_{12}(t')\|_{\Omega,2}\|\tau_{12}(t')\|_{\Omega,2} + \|\tau_{12}(t')\|^2_{\Omega,2} + \|u_{12}(t')\|_{H^1_2(\Omega)}\|\tau_{12}(t')\|_{\Omega,2}) \\ &\le \frac{1}{2}\|u_{12}(t')\|^2_{H^1_2(\Omega)} + C\|\tau_{12}(t')\|^2_{\Omega,2}.\end{aligned}$$

Integrating over $(0,t)$, we can assert that

$$\|\tau_{12}(t)\|_{\Omega,2}^2 \le \frac{1}{2}\|u_{12}\|^2_{L_2(0,t;H_2^1(\Omega))} + C\|\tau_{12}\|^2_{t,\Omega,2,2}. \tag{2.18}$$

Adding equation (2.16) and (2.18) as well as absorbing the term $\frac{1}{2}\|u_{12}\|^2_{L_2(0,t;H_2^1(\Omega))}$ yields

$$\|u_{12}(t)\|_{\Omega,2}^2 + \|\tau_{12}(t)\|_{\Omega,2}^2 \le C\int_0^t \|u_{12}(t')\|_{\Omega,2}^2 + \|\tau_{12}(t')\|_{\Omega,2}^2 dt',$$

and hence, $u_{12} = 0$ and $\tau_{12} = 0$ by Gronwall's Lemma (see Proposition 1.21). This completes the proof. □

2.2 Generalized viscoelastic fluids and Oldroyd-B fluids on unbounded domains with Dirichlet boundary conditions

The aim of this section is to consider (2.1) in unbounded domains $\Omega \subset \mathbb{R}^n$. The only assumption on the domain is, that for $q > n$ the Helmholtz decomposition exists for $L_r(\Omega)$, $r \in \{q, \frac{q}{q-1}\}$, and that for $\lambda \ge 0$ a shift of the Stokes operator $\lambda + A_r$, $r \in \{q, \frac{q}{q-1}\}$, admits bounded imaginary powers with a power angle less than $\frac{\pi}{2}$. Examples of such domains are exterior domains, layers, half spaces, and the whole space. We will restrict our attention in the analysis of (2.1) to the case that $\alpha > 0$ is a constant, $g(0,0) = 0$, and $\Gamma_S = \emptyset$, i.e. we investigate

$$\left\{\begin{array}{rcll} \rho(\partial_t u + u.\nabla u) - \alpha\Delta u + \nabla\pi &=& \operatorname{Div}\mu(\tau) + f & \text{in } (0,T_0)\times\Omega, \\ \operatorname{div} u &=& 0 & \text{in } (0,T_0)\times\Omega, \\ \partial_t\tau + u.\nabla\tau &=& g(\nabla u,\tau) & \text{in } (0,T_0)\times\Omega, \\ u &=& 0 & \text{on } (0,T_0)\times\partial\Omega, \\ u(0) &=& u_0 & \text{in } \Omega, \\ \tau(0) &=& \tau_0 & \text{in } \Omega, \end{array}\right. \tag{2.19}$$

with the initial values $(u_0,\tau_0) \in (L_{p,\sigma}(\Omega), D(A_q))_{1-\frac{1}{p},p} \times H_q^1(\Omega)$.

We show in this section local-in-time existence of (2.19) in the L_p-L_q-setting, $\frac{1}{p} + \frac{n}{2q} < \frac{1}{2}$, more precisely, we prove the existence of a unique strong solution in the regularity class

$$\begin{aligned} &u \in H_p^1(0,T;L_{q,\sigma}(\Omega)) \cap L_p(0,T;D(A_q)), \quad \pi \in L_p(0,T;\widehat{H}_q^1(\Omega)), \\ &\text{and} \quad \tau \in W_\infty^1(0,T;L_q(\Omega)) \cap L_\infty(0,T;H_q^1(\Omega)). \end{aligned}$$

In the case of an Oldroyd-B fluid, where the special form (see (2.3)) of g and μ is assumed, we can prove basically the same result for more values of p and q, more precisely for $1 < p < \infty$ with $p \neq 2$ and $n < q < \infty$. In this case, the elastic part of the stress only satisfies

$$\tau \in H_p^1(0,T;L_q(\Omega)) \cap L_\infty(0,T;H_q^1(\Omega)).$$

Let us state the first main result of this section.

Theorem 2.6. *Fix $n \in \mathbb{N}$, $n \ge 2$, $1 < p,q,q' < \infty$ with $\frac{1}{p} + \frac{n}{2q} < \frac{1}{2}$ and $\frac{1}{q} = \frac{1}{q'} = 1$, as well as $T_0,\rho,\alpha > 0$. Let $r \in \{q,q'\}$, $\lambda \ge 0$, and let $\Omega \subset \mathbb{R}^n$ be a domain with a uniform C^2-boundary, such* *that the Helmholtz decomposition exists for $L_r(\Omega)$ and $\lambda + A_r$ admits bounded imaginary powers with a power angle less than $\frac{\pi}{2}$. Moreover, we assume*

$$\mu \in C^1(\mathbb{R}^{n\times n}, \mathbb{R}^{n\times n}) \quad \text{and} \quad g \in C^1(\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}, \mathbb{R}^{n\times n}) \quad \text{with} \quad g(0,0) = 0.$$

Then, for each $f \in L_p(0, T_0; L_q(\Omega))$ and $(u_0, \tau_0) \in (L_{q,\sigma}(\Omega), D(A_q))_{1-\frac{1}{p},p} \times H^1_q(\Omega)$, there exists a time $0 < T < T_0$ and unique strong solution (u, π, τ) of (2.19) on the time interval $(0, T)$ in the regularity class

$$\begin{aligned} &u \in H^1_p(0, T; L_{q,\sigma}(\Omega)) \cap L_p(0, T; D(A_q)), \quad \pi \in L_p(0, T; \widehat{H}^1_q(\Omega)), \\ &\text{and} \quad \tau \in W^1_\infty(0, T; L_q(\Omega)) \cap L_\infty(0, T; H^1_q(\Omega)). \end{aligned}$$

A corresponding result about solutions on arbitrary finite time intervals $(0, T_0)$ for sufficient small initial data can be found in [GGN12].

In the case of an Oldroyd-B fluid, where in addition the special form (see (2.3)) of g and $\mu(\tau) = \mu\tau$ is assumed, we can extend the range of p and q, i.e. $1 < p < \infty$ with $p \neq 2$ and $n < q < \infty$. For the Oldroyd-B model

$$\left\{ \begin{array}{rcll} \rho(\partial_t u + u.\nabla u) - \alpha\Delta u + \nabla\pi &=& \mu \operatorname{Div}\tau + f & \text{in } (0,T_0)\times\Omega, \\ \operatorname{div} u &=& 0 & \text{in } (0,T_0)\times\Omega, \\ \partial_t\tau + u.\nabla\tau + \beta\tau &=& \gamma E u + \delta((\nabla u)^T\tau + \tau\nabla u) & \text{in } (0,T_0)\times\Omega, \\ u &=& 0 & \text{on } (0,T_0)\times\partial\Omega, \\ u(0) &=& u_0 & \text{in } \Omega, \\ \tau(0) &=& \tau_0 & \text{in } \Omega, \end{array} \right. \tag{2.20}$$

we prove the second main theorem of this section.

Theorem 2.7. *Fix $n \in \mathbb{N}$, $n \geq 2$, $1 < p, q, q' < \infty$, with $p \neq 2$, $q > n$, and $\frac{1}{q} + \frac{1}{q'} = 1$, as well as $T_0, \rho, \alpha > 0$, and $\beta, \gamma, \delta, \mu \in \mathbb{R}$. Let $r \in \{q, q'\}$, $\lambda \geq 0$, and let $\Omega \subset \mathbb{R}^n$ be a domain with a uniform C^2-boundary, such that the Helmholtz decomposition exists of $L_r(\Omega)$ and $\lambda + A_r$ admits bounded imaginary powers with a power angle less than $\frac{\pi}{2}$. Then, for each $f \in L_p(0, T_0; L_q(\Omega))$ and $(u_0, \tau_0) \in (L_{q,\sigma}(\Omega), D(A_q))_{1-\frac{1}{p},p} \times H^1_q(\Omega)$, there exists a time $0 < T < T_0$ and unique strong solution (u, π, τ) of (2.20) on the time interval $(0, T)$ in the regularity class*

$$\begin{aligned} &u \in H^1_p(0, T; L_{q,\sigma}(\Omega)) \cap L_p(0, T; D(A_q)), \quad \pi \in L_p(0, T; \widehat{H}^1_q(\Omega)), \\ &\text{and} \quad \tau \in H^1_p(0, T; L_q(\Omega)) \cap L_\infty(0, T; H^1_q(\Omega)). \end{aligned}$$

Remark 2.8. In the case that Ω is a bounded domain, an exterior domain with $n \geq 3$, or a half space and $1 < p < \infty$ and $n < q < \infty$ with $\frac{1}{p} + \frac{1}{2q} < 1$, the space for the initial value in the previous two theorems can be characterized more explicitly (see (1.3))

$$(L_{q,\sigma}, D(A_q))_{1-\frac{1}{p},p} = \{u \in B^{2-\frac{2}{p}}_{q,p}(\Omega) \colon \operatorname{div} u = 0,\, u|_{\partial\Omega} = 0\}.$$

But in general, we only have

$$(L_{q,\sigma}, D(A_q))_{1-\frac{1}{p},p} \subset \{u \in B^{2-\frac{2}{p}}_{q,p}(\Omega) \colon \operatorname{div} u = 0,\, u|_{\partial\Omega} = 0\}$$

for uniformly C^2-domains. Examples, where this inclusion is strict are aperture domains.

Remark 2.9. The main theorems, Theorem 2.6 and Theorem 2.7 cover the whole space, the half space, bent half spaces, layers, bent layers, bounded and exterior domains with a uniform C^2-boundary, since the Stokes operator admits bounded imaginary powers (see Giga [Gig85], Noll and Saal [NS03], as well as Abels and Terasawa [AT09]).

Sketch of the proof

The methods used in the previous section are not applicable here, since the domain is unbounded and hence, the compact embeddings fail. Also the standard contraction mapping principle is not directly applicable. Tyring to show the contraction property, we deduce that the difference of two solutions $\tau_2 - \tau_1$ fulfill a transport equation, where the term

$$(\tilde{w}_2 - \tilde{w}_1).\nabla\tau_2 \tag{2.21}$$

appears on the right-hand side of this transport equation (see (2.32)). This term cannot be controlled in $L_1(0,T;H^1_q(\Omega))$, due to lack of regularity of τ_2 (see Proposition 1.10). To avoid this problem, we apply modified version of the contraction mapping principle (Proposition 1.13), where is is sufficient to show the contraction in a weaker topology.

More precisely, we consider basically the same fixed point map Φ as in the previous section and we apply a modified version of the contraction mapping principle (Proposition 1.13) in the setting

$$X = H^1_p(0,T;L_{q,\sigma}(\Omega)) \cap L_p(0,T;D(A_q)) \times L_\infty(0,T;H^1_q(\Omega)),$$
$$X^w = H^{\frac{1}{2}}_p(0,T;L_q(\Omega)) \cap L_p(0,T;H^1_q(\Omega)) \times L_\infty(0,T;L_q(\Omega)).$$

The aim is to show, that we can choose a closed, convex, and bounded subset $\mathcal{K} \subset X$, such that $\Phi(\mathcal{K}) \subset \mathcal{K}$ and $\Phi|_{\mathcal{K}}$ is a contraction in the topology of X^w.

To prove the contraction property in X^w, we need estimates of the solution of the associated linearisation in X^w. In the proposition on the transport equation (Proposition 1.10), we proved a-priori estimates for the solution of the transport equation in $X^w_2 = L_\infty(0,T;L_q(\Omega))$, provided the right-hand side belongs to $L_1(0,T;L_q(\Omega))$. The advantage in showing the contraction in a weaker topology is, that the term $(\tilde{w}_2 - \tilde{w}_1).\nabla\tau_2$ (see (2.21)), which causes the difficulties in showing the contraction in X, can be estimated in $L_1(0,T;L_q(\Omega))$. This way, we can prove an estimate for the difference $\tau_2 - \tau_1$ in X^w_2. Further, to show the contraction in X^w, we estimate the solution of the Stokes problem in the space $X^w_1 = H^{\frac{1}{2}}_p(0,T;L_q(\Omega)) \cap L_p(0,T;H^1_q(\Omega))$, provided the right-hand side is of the form $\operatorname{Div} F$, $F \in L_p(0,T;H^1_q(\Omega))$. To prove this estimate, we use that a shift of the Stokes operator admits bounded imaginary powers. The assumption, that the viscosity function is constant, plays an important role in the prove of the estimate in X^w_1.

2.2.1 An L_p-L_q-estimate for the Stokes problem

In this subsection, we prove that the solution $u \in H^1_p(0,T;L_{q,\sigma}(\Omega)) \cap L_p(0,T;D(A_q))$ of the Stokes system

$$\begin{cases} \rho\partial_t u - \alpha\Delta u + \nabla\pi &= \operatorname{Div} F \quad \text{in } (0,T)\times\Omega, \\ \operatorname{div} u &= 0 \quad \text{in } (0,T)\times\Omega, \\ u &= 0 \quad \text{on } (0,T)\times\partial\Omega, \\ u(0) &= 0 \quad \text{in } \Omega, \end{cases} \tag{2.22}$$

with zero initial value and right-hand side $\operatorname{Div} F$, $F \in L_p(0,T;H^1_q(\Omega))$, can be estimated by

$$\|u\|_{H_p^{\frac{1}{2}}(0,T;L_q(\Omega))\cap L_p(0,T;H_q^1(\Omega))} \le C\|F\|_{L_p(0,T;L_q(\Omega))}, \tag{2.23}$$

provided that $\Omega \subset \mathbb{R}^n$ is a domain with a uniform C^2-boundary, such that the Helmholtz decomposition exists for $L_q(\Omega)$ and $L_{q'}(\Omega)$, $q' = \frac{q}{q-1}$, as well as that the Stokes operators A_q and $A_{q'}$, after a shift, admit bounded imaginary powers with a power angle less than $\frac{\pi}{2}$.

The next proposition is a variant of Giga, Giga, and Sohr [GGS93, Corollary 4.2].

Proposition 2.10. *Fix $n \in \mathbb{N}$, $n \ge 2$, $1 < p,q,q' < \infty$, with $\frac{1}{q} + \frac{1}{q'} = 1$ and $p \neq 2$, as well as $T_0, \rho, \alpha > 0$. Let $r \in \{q, q'\}$, $\lambda \ge 0$, and let $\Omega \subset \mathbb{R}^n$ be a domain with a uniformly C^2-boundary, such that the Helmholtz decomposition exists for $L_r(\Omega)$ and $\lambda + A_r$ admits bounded imaginary powers with a power angle less than $\frac{\pi}{2}$. Then, there exists a constant $C > 0$, such that for each $0 < T < T_0$ and right-hand side $\operatorname{Div} F$, with $F \in L_p(0,T;H^1_q(\Omega))$, the unique solution $u \in {}_0H^1_p(0,T;L_{q,\sigma}(\Omega)) \cap L_p(0,T;D(A_q))$ of the Stokes problem with zero initial value (2.22) can be estimated by*

$$\|u\|_{{}_0H_p^{\frac{1}{2}}(0,T;L_q(\Omega))\cap L_p(0,T;H_q^1(\Omega))} \le C\|F\|_{T,\Omega,p,q}.$$

Proof. There is no loss of generality in assuming $\alpha = \rho = 1$. With C, we always denote a generic constant, which may change from line to line, but is always independent of F and T, $0 < T < T_0$. Let $F \in L_p(0,T;H^1_q(\Omega))$. Applying the Helmholtz projection P_q to (2.22), we obtain the equivalent problem

$$u' + A_q u = P_q \operatorname{Div} F \quad \text{in } (0,T), \qquad u(0) = 0. \tag{2.24}$$

The existence of a unique solution $u \in H^1_p(0,T;L_{q,\sigma}(\Omega)) \cap L_p(0,T;D(A_q))$ of (2.24) follows from Remark 1.6, since a shift of the Stokes operator A_q admits bounded imaginary powers, with power angle less than $\frac{\pi}{2}$ and $P_q \operatorname{Div} F \in L_p(0,T;L_{q,\sigma}(\Omega))$. Next, we prove that this solution also satisfies the estimate

$$\|u\|_{{}_0H_p^{\frac{1}{2}}(0,T;L_q(\Omega))\cap L_p(0,T;H_q^1(\Omega))} \le C\|F\|_{T,\Omega,p,q}.$$

Fix $\lambda_0 > \lambda$ with $\lambda_0 \in \rho(-A_q)$ and set $w := (\lambda_0 + A_q)^{-\frac{1}{2}}u$. Then, since

$$w \in H^1_p(0,T;L_{q,\sigma}(\Omega)) \cap L_p(0,T;D(A_q)),$$

we compute

$$(\frac{d}{dt} + A_q)w = (\frac{d}{dt} + A_q)(\lambda_0 + A_q)^{-\frac{1}{2}}u = (\lambda_0 + A_q)^{-\frac{1}{2}}(\frac{d}{dt} + A_q)u = (\lambda_0 + A_q)^{-\frac{1}{2}}P_q \operatorname{Div} F,$$

and

$$w(0) = ((\lambda_0 + A_q)^{-\frac{1}{2}}u)(0) = (\lambda_0 + A_q)^{-\frac{1}{2}}u(0) = 0.$$

Thus, w is the solution of

$$w' + A_q w = (\lambda_0 + A_q)^{-\frac{1}{2}}P_q \operatorname{Div} F \quad \text{in } (0,T), \quad w(0) = 0. \tag{2.25}$$

Next, we show that the right-hand side of (2.25) satisfies

$$\|(\lambda_0 + A_q)^{-\frac{1}{2}} P_q \operatorname{Div} F\|_{T,\Omega,p,q} \le C\|F\|_{T,\Omega,p,q},$$

by a duality argument. Let $1 < q' < \infty$ with $\frac{1}{q'} + \frac{1}{q} = 1$. We have

$$(u, A_q v)_\Omega = -(u, \Delta_D v)_\Omega = -(\Delta_D u, v)_\Omega = (A_{q'} u, v)_\Omega, \quad (u, v) \in D(A_{q'}) \times D(A_q),$$

where Δ_D denotes the Laplace operator with Dirichlet boundary conditions, and hence, we conclude that $(A_q)' \supset A_{q'}$. Since $(A_q)'$ is densely defined, $(A_q)'$ generates an analytic semigroup and the resolvent of $(A_q)'$ and $A_{q'}$ have a nonempty intersection, and hence $(A_q)' = A_{q'}$. For $g \in L_{p'}(0,T; L_{q',\sigma}(\Omega))$ with $1 < p' < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, it follows that

(2.26)
$$\begin{aligned}\big|\big((\lambda_0 + A_q)^{-\frac{1}{2}} P_q \operatorname{Div} F \big| g\big)_{T,\Omega}\big| &= \big|\big(\operatorname{Div} F \big| (\lambda_0 + A_{q'})^{-\frac{1}{2}} g\big)_{T,\Omega}\big| = \big|\big(F : \nabla(\lambda_0 + A_{q'})^{-\frac{1}{2}} g\big)_{T,\Omega}\big| \\ &\le \|F\|_{T,\Omega,p,q} \|\nabla(\lambda_0 + A_{q'})^{-\frac{1}{2}} g\|_{T,\Omega,p',q'} \le C\|F\|_{T,\Omega,p,q}\|g\|_{T,\Omega,p',q'},\end{aligned}$$

where we used the Dirichlet boundary condition of $(\lambda_0 + A_{q'})^{-\frac{1}{2}} g$ and the continuity of the operator

$$(\lambda_0 + A_{q'})^{-\frac{1}{2}} \in \mathcal{L}\big(L_{p'}(0,T; L_{q',\sigma}(\Omega)), L_{p'}(0,T; H^1_{q',0}(\Omega) \cap L_{q',\sigma}(\Omega))\big),$$

which follows by $D(A_{q'}^{\frac{1}{2}}) = [L_{q',\sigma}(\Omega), D(A_{q'})]_{\frac{1}{2}} = H^1_{q',0}(\Omega) \cap L_{q',\sigma}(\Omega)$ (see Lemma 1.7). Since w is the solution of the Stokes problem (2.25) with right-hand side

$$(\lambda_0 + A_q)^{-\frac{1}{2}} P_q \operatorname{Div} F \in L_p(0,T; L_{q,\sigma}(\Omega)),$$

it follows that

$$\|w\|_{{}_0H^1_p(0,T;L_q(\Omega)) \cap L_p(0,T;D(A_q))} \le C\|(\lambda_0 + A_q)^{-\frac{1}{2}} P_q \operatorname{Div} F\|_{T,\Omega,p,q} \le C\|F\|_{T,\Omega,p,q}, \tag{2.27}$$

by the maximal regularity of the Stokes operator and (2.26). Taking into account [Prü02, Corollary 2.2], we deduce the continuity of the operator

(2.28) $(\lambda_0 + A_q)^{\frac{1}{2}}$
$$\in \mathcal{L}\big({}_0H^1_p(0,T; L_{q,\sigma}(\Omega)) \cap L_p(0,T; D(A_q)), {}_0H^{\frac{1}{2}}_p(0,T; L_{q,\sigma}(\Omega)) \cap L_p(0,T; D(A_q^{\frac{1}{2}}))\big).$$

Combining (2.27) and (2.28), we conclude that

$$\begin{aligned}\|u\|_{{}_0H^{\frac{1}{2}}_p(0,T;L_q(\Omega)) \cap L_p(0,T;H^1_q(\Omega))} &= \|(\lambda_0 + A_q)^{\frac{1}{2}} w\|_{{}_0H^{\frac{1}{2}}_p(0,T;L_q(\Omega)) \cap L_p(0,T;H^1_q(\Omega))} \\ &\le C\|w\|_{{}_0H^1_p(0,T;L_{q,\sigma}(\Omega)) \cap L_p(0,T;D(A_q))} \\ &\le C\|F\|_{T,\Omega,p,q}.\end{aligned}$$

□

We are now in a position to prove the main theorems.

2.2.2 Proof of the main theorems

In this section, we give a proof of both main theorems, Theorem 2.6 and Theorem 2.7.

Proof of Theorem 2.6 and 2.7. As in the proof of Theorem 2.1, we reduce (2.19) to $u_0 = 0$ and $f = 0$ and rewrite it in form of a fixed point equation.

Reduction to $u_* = 0$ and $f = 0$ and Fixed point formulation

We proceed exactly the same way as in the proof of Theorem 2.1. The situation here is less complex, since $\alpha > 0$ is constant and $\Gamma_S = \emptyset$. However, in order to apply Proposition 2.10, we need a representation of the right-hand side in divergence form. We choose

$$(u_*, \pi_*) \in H^1_p(0,T; L_{q,\sigma}(\Omega)) \cap L_p(0,T; D(A_q)) \times L_p(0,T; \widehat{H}^1_q(\Omega))$$

to be the solution of the Stokes problem

$$\begin{cases} \rho\partial_t u_* - \alpha\Delta u_* + \nabla\pi_* &= f & \text{in } (0,T_0)\times\Omega, \\ \operatorname{div} u_* &= 0 & \text{in } (0,T_0)\times\Omega, \\ u_* &= 0 & \text{on } (0,T_0)\times\partial\Omega, \\ u_*(0) &= u_0 & \text{in } \Omega, \end{cases}$$

given by Proposition 2.10, and we set

$$u = w + u_* \quad \text{and} \quad \pi = \pi_* + \psi.$$

Then, (u,π,τ) solves (2.19) if and only if (w,ψ,τ) solves

$$(2.29)\qquad \begin{cases} \rho\partial_t w - \alpha\Delta w + \nabla\psi &= f_* + \operatorname{Div}(F^D_w(w) + F^D_\tau(\tau)) & \text{in } (0,T_0)\times\Omega, \\ \operatorname{div} w &= 0 & \text{in } (0,T_0)\times\Omega, \\ \partial_t\tau + (w+u_*).\nabla\tau &= G(w,\tau) & \text{in } (0,T_0)\times\Omega, \\ w &= 0 & \text{on } (0,T_0)\times\partial\Omega, \\ w(0) &= 0 & \text{in } \Omega, \\ \tau(0) &= \tau_0 & \text{in } \Omega, \end{cases}$$

where the terms on the right-hand side of the Stokes equation f_*, F^D_w, and F^D_τ and the right-hand side of the transport equation G are defined below. For $\tilde f, \tilde g \in H^1_q(\Omega)$ with $\operatorname{div}\tilde f = 0$, it holds that $\tilde f.\nabla\tilde g = \operatorname{Div}(\tilde g \otimes \tilde f)$. Hence, we obtain

$$f_* = -\rho u_*.\nabla u_* + \operatorname{Div}\mu(\tau_0) \in L_p(0,T; L_q(\Omega)), \quad F^D_w(w) := -\rho u_* \otimes w - \rho w \otimes u_* - \rho w \otimes w$$

and

$$F^D_\tau(\tau) := \mu(\tau) - \mu(\tau_0).$$

Compared to the previous section, the representation of the function f_* simplifies and we have $\operatorname{Div} F^D_w(w) = F_w(w)$ and $\operatorname{Div} F^D_\tau(\tau) = F_\tau(\tau)$, where F_w and F_τ are defined in the previous section, since $\alpha > 0$ is constant. Further, the right-hand side of the transport equation is, as in the previous section, given by

$$G(w,\tau) = g(\nabla(w+u_*),\tau).$$

Problem (2.29) exactly match (2.6) in Section 2.1 in the case $\alpha > 0$ is constant and $\Gamma_S = \emptyset$.

We formulate (2.29) in the form of a fixed point equation in a suitable Banach space. To prove Theorem 2.6 it would be possible to choose basically the same spaces as in Section 2.1. But, in the case $1 < p < \infty$ and $n < q < \infty$ of Theorem 2.7, it is convenient to modify the solution space for the elastic part of the transport equation. The new space for the elastic part of the stress admits a weaker topology, but this topology is sufficient to treat the nonlinearities, since no nonlinearities appear on the boundary. For $1 < p < \infty$ and $n < q < \infty$, we recall the definition of the solution space for the velocity field ${}_0\mathbb{E}_u(T,\Omega)$ and define the solution space for the elastic part of the stress:

$$\begin{array}{lcl} {}_0\mathbb{E}^{p,q}_{u,c}(T,\Omega) & := & {}_0H^1_p(0,T;L_{q,\sigma}(\Omega)) \cap L_p(0,T;D(A_q)), \\ \mathbb{E}^{\#}_{\tau}(T,\Omega) & := & L_\infty(0,T;H^1_q(\Omega)). \end{array}$$

Furthermore, we recall the definition of the solution space for the velocity field, where no initial value is prescribed:

$$\mathbb{E}^{p,q}_u(T,\Omega) \;=\; H^1_p(0,T;L_q(\Omega)) \cap L_p(0,T;H^2_q(\Omega)).$$

Moreover, we define the space for the data

$$\begin{array}{lcl} \mathbb{F}^{p,q}_f(T,\Omega) & := & L_p(0,T;L_q(\Omega)), \\ \mathbb{G}^{\#}(T,\Omega) & := & L_1(0,T;H^1_q(\Omega)). \end{array}$$

The space $\mathbb{E}^{p,q}_u(T,\Omega)$ was already defined in the preliminaries. Compared to Section 2.1, we can choose a space with a weaker topology $\mathbb{G}^{\#}(T,\Omega)$ for the right-hand sides of the transport equation, since we need less regularity for the elastic part of the stress.

Similar as in the previous section, problem (2.29) can be rewritten as a fixed point problem of the map

$$\begin{array}{rcl} \Phi\colon {}_0\mathbb{E}^{p,q}_{u,c}(T,\Omega) \times \mathbb{E}^{\#}_{\tau}(T,\Omega) & \to & {}_0\mathbb{E}^{p,q}_{u,c}(T,\Omega) \times \mathbb{E}^{\#}_{\tau}(T,\Omega), \\ (w,\tau) & \mapsto & \tilde{\Phi}_{0,\tau_0}(w, f_* + \operatorname{Div}(F^D_w(w) + F^D_\tau(\tau)), G(w,\tau)), \end{array} \tag{2.30}$$

where

$$\begin{array}{rcl} \tilde{\Phi}_{0,\tau_0}\colon {}_0\mathbb{E}^{p,q}_{u,c}(T,\Omega) \times \mathbb{F}^{p,q}_f(T,\Omega) \times \mathbb{G}^{\#}(T,\Omega) & \to & {}_0\mathbb{E}^{p,q}_{u,c}(T,\Omega) \times \mathbb{E}^{\#}_{\tau}(T,\Omega), \\ (\tilde{w}, \tilde{f}, \tilde{g}) & \mapsto & (w,\tau) \end{array}$$

denotes the solution operator to the following problem:

$$\left\{ \begin{array}{rcll} \rho\partial_t w - \alpha\Delta w + \nabla\psi & = & \tilde{f} & \text{in } (0,T_0)\times\Omega, \\ \operatorname{div} w & = & 0 & \text{in } (0,T_0)\times\Omega, \\ \partial_t\tau + (\tilde{w} + u_*).\nabla\tau & = & \tilde{g} & \text{in } (0,T_0)\times\Omega, \\ w & = & 0 & \text{on } (0,T_0)\times\partial\Omega, \\ w(0) & = & 0 & \text{in } \Omega, \\ \tau(0) & = & \tau_0 & \text{in } \Omega. \end{array} \right. \tag{2.31}$$

The main difference to Section 2.1 is, that we consider the case $p \neq q$ and that we replace $(\mathbb{E}_\tau(T,\Omega), \mathbb{G}(T,\Omega))$ by $(\mathbb{E}^{\#}_{\tau}(T,\Omega), \mathbb{G}^{\#}(T,\Omega))$. Therefore, we denote the fixed point map Φ, the solution operator for the linearized problem $\tilde{\Phi}_{0,\tau_0}$, and the nonlinearity on the right-hand of the transport equation G in the same way as in the previous section.

The maps Φ and $\tilde{\Phi}_{0,\tau_0}$ are well-defined: The Stokes equation and the transport equation are decoupled in equation (2.31) and can be solved separately. We can solve the Stokes problem due to Proposition 2.10, and the transport equation due to the proposition on the transport equation (Proposition 1.10). This shows that $\tilde{\Phi}_{0,\tau_0}$ is well-defined. In Lemma 2.11 and 2.12 we show the mapping properties:

$$\begin{aligned}
&\operatorname{Div} F_w^D \colon {}_0\mathbb{E}_{u,c}^{p,q}(T,\Omega) \to \mathbb{F}_f^{p,q}(T,\Omega), \quad \operatorname{Div} F_\tau^D \colon \mathbb{E}_\tau^{\#}(T,\Omega) \to \mathbb{F}_f^{p,q}(T,\Omega),\\
&G \colon {}_0\mathbb{E}_{u,c}^{p,q}(T,\Omega) \times \mathbb{E}_\tau^{\#}(T,\Omega) \to \mathbb{G}^{\#}(T,\Omega) \cap L_p(0,T;L_q(\Omega)).
\end{aligned}$$

This implies, that Φ is well-defined.

Analysis of Φ

The aim is to show that Φ admits a unique fixed point. For $0 < R_1, R_2 < \infty$ and $0 < T < T_0$, we define the balls

$$\begin{aligned}
\mathcal{K}_w(T,R_1) &:= \{w \in {}_0\mathbb{E}_{u,c}^{p,q}(T,\Omega) \colon \|w\|_{{}_0\mathbb{E}_u^{p,q}(T,\Omega)} \le R_1\},\\
\mathcal{K}_\tau^{\#}(T,R_2) &:= \{\tau \in \mathbb{E}_\tau^{\#}(T,\Omega) \colon \tau(0) = \tau_0 \text{ and } \|\tau\|_{\mathbb{E}_\tau^{\#}(T,\Omega)} \le R_2\},\\
\mathcal{K}^{\#}(T,R_1,R_2) &:= \mathcal{K}_w(T,R_1) \times \mathcal{K}_\tau^{\#}(T,R_2).
\end{aligned}$$

The map Φ maps $\mathcal{K}^{\#}(T,R_1,R_2)$ into itself

We show, that we can choose $T, R_1, R_2 > 0$, such that

$$\Phi(\mathcal{K}^{\#}(T,R_1,R_2)) \subset \mathcal{K}^{\#}(T,R_1,R_2).$$

To prove this relation, a proper understanding of the nonlinearities is necessary. We analyse these in the following two lemmas. In the first lemma, we consider the Oldroyd-B case, where a special form of g and μ is assumed. Thanks to this special form, we can estimate the nonlinearities for $1 < p < \infty$ and $n < q < \infty$. In the second lemma, we consider more general functions g and μ. To estimate the nonlinearities in this case, stricter conditions on p and q are needed. We need at least that $X_u^{p,q}(T,\Omega) \hookrightarrow L_\infty(0,T;L_\infty(\Omega))$, and therefore $\frac{1}{p} + \frac{n}{2q} < \frac{1}{2}$.

In the Oldroyd-B case, the following Lemma holds.

Lemma 2.11. *Fix $n \in \mathbb{N}$, $n \ge 2$, $T_0, R_0, R_2 > 0$, $1 < p < \infty$, and $n < q < \infty$. Let $\Omega \subset \mathbb{R}^n$ be a domain with a uniform C^2-boundary. Assume that G has the special form*

$$G(w,\tau) = -\beta\tau + \gamma E(w+u_*) + \delta((\nabla(w+u_*))^T\tau + \tau\nabla(w+u_*)),$$

with $\beta, \gamma, \delta \in \mathbb{R}$, and that $\mu \in \mathbb{R}$ is constant. Then, there exists a constant $C > 0$ and a function $O \colon \mathbb{R}_+ \to \mathbb{R}_+$ with $O(t) \to 0$ for $t \to 0$, such that for all $R_1 \in (0,R_0)$, $T \in (0,T_0)$, and $(w,\tau) \in \mathcal{K}^{\#}(T,R_1,R_2)$ the estimates

$$\begin{aligned}
\|\operatorname{Div} F_w^D(w)\|_{\mathbb{F}_f^{p,q}(T,\Omega)} &\le CR_1^2 + O(T),\\
\|\operatorname{Div} F_\tau^D(\tau)\|_{\mathbb{F}_f^{p,q}(T,\Omega)} &\le O(T),\\
\|G(w,\tau)\|_{\mathbb{G}^{\#}(T,\Omega)} &\le O(T),\\
\|G(w,\tau)\|_{T,\Omega,p,q} &\le C
\end{aligned}$$

hold.

Proof. Let $0 < R_0, R_2, T_0$, $0 < R_1 < R_0$, $0 < T < T_0$, and $(w,\tau) \in \mathcal{K}(T, R_1, R_2)$. We denote by C a generic constant and by $O\colon \mathbb{R}_+ \to \mathbb{R}_+$ a generic function, with $O(t) \to 0$ for $t \to 0$, which may change from line to line, but is always independent of T, R_1, w, and τ.

We recall the definition $F_w^D(w) = -\rho u_* \otimes w - \rho w \otimes u_* - \rho w \otimes w$, the representation

$$\operatorname{Div} F_w^D = -\rho u_*.\nabla w - \rho w.\nabla u_* - \rho w.\nabla w,$$

and the embedding

$${}_0\mathbb{E}_u^{p,q}(T,\Omega) \hookrightarrow L_{3p}(0,T;L_{3q}(\Omega)) \cap L_{\frac{3p}{2}}(0,T;H^1_{\frac{3q}{2}}(\Omega))$$

with embedding constant independent of $0 < T < T_0$ (Proposition 1.14). Hence

$$\begin{aligned}
\|\operatorname{Div} F_w^D(w)\|_{T,\Omega,p,q} &\le \rho\|u_*.\nabla w\|_{T,\Omega,p,q} + \rho\|w\nabla u_*\|_{T,\Omega,p,q} + \rho\|w.\nabla w\|_{T,\Omega,p,q}\\
&\le C\big(\|u_*\|_{T,\Omega,3p,3q}\|w\|_{L_{\frac{3p}{2}}(0,T;H^1_{\frac{3q}{2}}(\Omega))} + \|w\|_{T,\Omega,3p,3q}\|u_*\|_{L_{\frac{3p}{2}}(0,T;H^1_{\frac{3q}{2}}(\Omega))}\\
&\quad + \|w\|_{T,\Omega,3p,3q}\|w\|_{L_{\frac{3p}{2}}(0,T;H^1_{\frac{3q}{2}}(\Omega))}\big)\\
&\le C\big((\|u_*\|_{T,\Omega,3p,3q} + \|u_*\|_{L_{\frac{3p}{2}}(0,T;H^1_{\frac{3q}{2}}(\Omega))})\|w\|_{{}_0\mathbb{E}_u^{p,q}(T,\Omega)} + \|w\|^2_{{}_0\mathbb{E}_u^{p,q}(T,\Omega)}\big)\\
&\le CR_1^2 + O(T).
\end{aligned}$$

Since $F_\tau^D = \mu(\tau - \tau_0)$, where $\mu \in \mathbb{R}$ is constant, it follows that

$$\begin{aligned}
\|\operatorname{Div} F_\tau^D(\tau)\|_{T,\Omega,p,q} &= |\mu|\|\operatorname{Div}(\tau - \tau_0)\|_{T,\Omega,p,q}\\
&\le |\mu|T^{\frac{1}{p}}(\|\nabla\tau\|_{T,\Omega,\infty,q} + \|\nabla\tau_0\|_{T,\Omega,\infty,q})\\
&\le |\mu|T^{\frac{1}{p}}(R_2 + \|\tau_0\|_{H^1_q(\Omega)})\\
&\le O(T).
\end{aligned}$$

The right-hand side of the transport equation is defined by

$$G(w,\tau) = -\beta\tau + \gamma E(w+u_*) + \delta((\nabla(w+u_*))^T\tau + \tau\nabla(w+u_*)).$$

For $r' \in \{1,p\}$, we deduce the estimate

$$\begin{aligned}
&\|G(w,\tau)\|_{T,\Omega,r',q}\\
&\quad\le |\beta|\|\tau\|_{T,\Omega,r',q} + |\gamma|\|\nabla(w+u_*)\|_{T,\Omega,r',q} + 2|\delta|\|\nabla(w+u_*)\|_{T,\Omega,r',q}\|\tau\|_{T,\Omega,\infty,\infty}\\
&\quad\le C\big(T^{\frac{1}{r'}}\|\tau\|_{T,\Omega,\infty,q} + T^{\frac{1}{r'}-\frac{1}{p}}\|\nabla(w+u_*)\|_{T,\Omega,p,q} + T^{\frac{1}{r'}-\frac{1}{p}}\|\nabla(w+u_*)\|_{T,\Omega,p,q}\|\tau\|_{T,\Omega,\infty,\infty}\big),
\end{aligned}$$

and hence $\|G(w,\tau)\|_{T,\Omega,p,q} \le C$ and $\|G(w,\tau)\|_{T,\Omega,1,q} \le O(T)$. Moreover, we deduce an estimate for the spatial derivative

$$\begin{aligned}
&\|\nabla G(w,\tau)\|_{T,\Omega,1,q}\\
&\quad\le |\beta|\|\nabla\tau\|_{T,\Omega,1,q} + |\gamma|\|\nabla^2(w+u_*)\|_{T,\Omega,1,q}\\
&\qquad + 2|\delta|\big(\|\nabla^2(w+u_*)\|_{T,\Omega,1,q}\|\tau\|_{T,\Omega,\infty,\infty} + \|\nabla(w+u_*)\|_{T,\Omega,1,\infty}\|\nabla\tau\|_{T,\Omega,\infty,q}\big)\\
&\quad\le C\big(T\|\tau\|_{\mathbb{E}_\tau^{\#}(T,\Omega)} + T^{1-\frac{1}{p}}\|w+u_*\|_{\mathbb{E}_u^{p,q}(T,\Omega)} + T^{1-\frac{1}{p}}\|w+u_*\|_{\mathbb{E}_u^{p,q}(T,\Omega)}\|\tau\|_{\mathbb{E}_\tau^{\#}(T,\Omega)}\big)\\
&\quad\le O(T).
\end{aligned}$$

□

In the general case, the following Lemma holds.

Lemma 2.12. *Fix $n \in \mathbb{N}$, $n \geq 2$, $T_0, R_0, R_2 > 0$, and $1 < p, q < \infty$ with $\frac{1}{p} + \frac{n}{2q} < \frac{1}{2}$. Let $\Omega \subset \mathbb{R}^n$ be a domain with a uniform C^2-boundary. Let $G(w, \tau) = g(\nabla(w + u_*), \tau)$,*

$$g \in C^1(\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}, \mathbb{R}^{n\times n}) \quad \textit{with} \quad g(0,0) = 0 \quad \textit{and} \quad \mu \in C^1(\mathbb{R}^{n\times n}, \mathbb{R}^{n\times n}).$$

Then, there exists a constant $C > 0$ and a function $O\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $O(t) \to 0$ for $t \to 0$, such that for all $R_1 \in (0, R_0)$, $T \in (0, T_0)$, and $(w, \tau) \in \mathcal{K}^\#(T, R_1, R_2)$ the estimates

$$\begin{array}{lcl} \|\operatorname{Div} F_w^D(w)\|_{\mathbb{F}_f^{p,q}(T,\Omega)} & \leq & CR_1^2 + O(T), \\ \|\operatorname{Div} F_\tau^D(\tau)\|_{\mathbb{F}_f^{p,q}(T,\Omega)} & \leq & O(T), \\ \|G(w,\tau)\|_{\mathbb{G}^\#(T,\Omega)} & \leq & O(T), \\ \|G(w,\tau)\|_{T,\Omega,\infty,q} & \leq & C \end{array}$$

hold.

Proof. The estimates of $\operatorname{Div} F_w^D(w)$ is proved in a more general setting in the previous lemma. Here, we only investigate $\operatorname{Div} F_\tau(\tau)$ and G.

Let $0 < R_0, R_2, T_0$, $0 < R_1 < R_0$, $0 < T < T_0$, and $(w, \tau) \in \mathcal{K}(T, R_1, R_2)$. We denote by C a generic constant and by $O\colon \mathbb{R}_+ \to \mathbb{R}_+$ a generic function, with $O(t) \to 0$ for $t \to 0$, which may change from line to line, but is always independent of T, R_1, w, and τ.

By the proposition on embedding theorems (Proposition 1.14) there exists a constant C_* with

$$\|w\|_{L_\infty(0,T;W^1_\infty(\Omega))\cap L_\infty(0,T;H^1_q(\Omega))} + \|u_*\|_{L_\infty(0,T;W^1_\infty(\Omega))\cap L_\infty(0,T;H^1_q(\Omega))} + \|\tau\|_{L_\infty(0,T;L_\infty(\Omega))} \leq C_*.$$

We recall the definition of the nonlinearity $F_\tau(\tau) = \mu(\tau) - \mu(\tau_0)$. It holds

$$\begin{aligned} \|\operatorname{Div} F_\tau^D(\tau)\|_{T,\Omega,p,q} &= \|\operatorname{Div}(\mu(\tau) - \mu(\tau_0))\|_{T,\Omega,p,q} \\ &\leq \sup_{|\eta|<C_*} |(\nabla\mu)(\eta)|(\|\nabla\tau\|_{T,\Omega,p,q} + \|\nabla\tau_0\|_{\Omega,q}) \\ &\leq \sup_{|\eta|<C_*} |(\nabla\mu)(\eta)|T^{\frac{1}{p}}(R_2 + \|\tau_0\|_{H^1_q(\Omega)}) \\ &\leq O(T). \end{aligned}$$

Let $r' \in \{1, \infty\}$. By the mean value theorem and $g(0,0) = 0$, it follows that

$$\begin{aligned} \|G(w,\tau)\|_{T,\Omega,r',q} &= \|g(\nabla(w+u_*),\tau)\|_{T,\Omega,r',q} \\ &= \|g(\nabla(w+u_*),\tau) - g(0,0)\|_{T,\Omega,r',q} \\ &\leq \sup_{|\eta_1|,|\eta_2|\leq C_*} |(\nabla g)(\eta_1,\eta_2)|\big(\|\nabla(w+u_*)\|_{T,\Omega,r',q} + \|\tau\|_{T,\Omega,r',q}\big) \\ &\leq \sup_{|\eta_1|,|\eta_2|\leq C_*} |(\nabla g)(\eta_1,\eta_2)|T^{\frac{1}{r'}}\big(\|\nabla(w+u_*)\|_{T,\Omega,\infty,q} + \|\tau\|_{T,\Omega,\infty,q}\big) \\ &\leq CT^{\frac{1}{r'}}\big(\|w\|_{L_\infty(0,T;H^1_q(\Omega))} + \|u_*\|_{L_\infty(0,T;H^1_q(\Omega))} + \|\tau\|_{T,\Omega,\infty,q}\big) \\ &\leq CT^{\frac{1}{r'}}. \end{aligned}$$

This shows $\|G(w,\tau)\|_{T,\Omega,\infty,q} \leq C$ and $\|G(w,\tau)\|_{T,\Omega,1,q} \leq O(T)$. The required estimate of the spatial derivative follows with the chain rule:

$$
\begin{aligned}
&\|\nabla G(w,\tau)\|_{T,\Omega,1,q} \\
&\quad \leq \sup_{|\eta_1|,|\eta_2|\leq C_*} |(\nabla g)(\eta_1,\eta_2)| \big(\|\nabla^2 (w+u_*)\|_{T,\Omega,1,q} + \|\nabla \tau\|_{T,\Omega,1,q} \big) \\
&\quad \leq \sup_{|\eta_1|,|\eta_2|\leq C_*} |(\nabla g)(\eta_1,\eta_2)| \big(T^{1-\frac{1}{p}} \|w\|_{{}_0\mathbb{E}_u^{p,q}(T,\Omega)} + T^{1-\frac{1}{p}} \|u_*\|_{\mathbb{E}_u^{p,q}(T,\Omega)} + T\|\tau\|_{\mathbb{E}_\tau^{\#}(T,\Omega)} \big) \\
&\quad \leq O(T).
\end{aligned}
$$

This shows $\|G(w,\tau)\|_{L_1(0,T;H_q^1(\Omega))} \leq O(T)$. □

Now, we show that we can choose T_0, R_0, R_2, $0 < T < T_0$ and $0 < R_1 < R_0$, such that

$$\Phi(\mathcal{K}^{\#}(T,R_1,R_2)) \subset \mathcal{K}^{\#}(T,R_1,R_2).$$

Set $R_0 = T_0 = 1$. Let $(w,\tau) = \Phi(\tilde{w},\tilde{\tau})$ with $(\tilde{w},\tilde{\tau}) \in \mathcal{K}^{\#}(T,R_1,R_2)$. By the maximal L_p-regularity of the Stokes operator (see Proposition 2.10), the proposition on the transport equation (Proposition 1.10), and the two previous lemmas (Lemma 2.11 and 2.12), we obtain in the case of an Oldroyd-B fluid and in the case of a generalized viscoelastic fluid

$$
\begin{aligned}
\|w\|_{{}_0\mathbb{E}_u^{p,q}(T,\Omega)} &= \|(\tilde{\Phi}_{0,\tau_0}(\tilde{w}, f_* + \operatorname{Div}(F_w^D(\tilde{w}) + F_\tau^D(\tilde{\tau})), G(\tilde{w},\tilde{\tau})))_1\|_{{}_0\mathbb{E}_u^{p,q}(T,\Omega)} \\
&\leq C\big(\|f_* + \operatorname{Div}(F_w^D(\tilde{w}) + F_\tau^D(\tilde{\tau}))\|_{\mathbb{F}_f^{p,q}(T,\Omega)} \big) \\
&\leq C\big(\|f_*\|_{\mathbb{F}_f^{p,q}(T,\Omega)} + \|\operatorname{Div} F_w(\tilde{w})\|_{\mathbb{F}_f^{p,q}(T,\Omega)} + \|\operatorname{Div} F_\tau(\tilde{\tau})\|_{\mathbb{F}_f^{p,q}(T,\Omega)} \big) \\
&\leq CR_1^2 + O(T),
\end{aligned}
$$

and

$$
\begin{aligned}
\|\tau\|_{\mathbb{E}_\tau^{\#}(T,\Omega)} &= \|(\tilde{\Phi}_{0,\tau_0}(w, f_* + F_w(w) + F_\tau(\tau), G(w,\tau)))_2\|_{\mathbb{E}_\tau(T,\Omega)} \\
&\leq C_{\mathrm{tra}}^{(1)} \big(\|\tau_0\|_{H_p^1(\Omega)} + \|G(w,\tau)\|_{\mathbb{G}^{\#}(T,\Omega)} \big) e^{C_{\mathrm{tra}}^{(1)} T^{1-\frac{1}{p}} \|w+u_*\|_{\mathbb{E}_u^{p,q}(T,\Omega)}} \\
&\leq C_{\mathrm{tra}}^{(1)} \big(\|\tau_0\|_{H_p^1(\Omega)} + O(T) \big) e^{O(T)},
\end{aligned}
$$

where $O\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a function with $O(t) \to 0$ for $t \to 0$, which is independent of R_1, $0 < R_1 < R_0$. Defining now $R_2 := 2C_{\mathrm{tra}}^{(1)} \|\tau_0\|_{H_p^1(\Omega)}$ and choosing first R_1, $0 < R_1 < R_0$ and then T, $0 < T < T_0$, sufficiently small, we find that $\Phi(\mathcal{K}^{\#}(T,R_1,R_2))$ is contained in $\mathcal{K}^{\#}(T,R_1,R_2)$.

The map Φ is a contraction

Next, we show that Φ is a contraction in the space $\mathbb{E}^w(T,\Omega) := \mathbb{E}_u^{p,q,w}(T,\Omega) \times \mathbb{E}_\tau^w(T,\Omega)$, with

$$
\begin{aligned}
\mathbb{E}_u^{p,q,w}(T,\Omega) &= H_p^{\frac{1}{2}}(0,T;L_q(\Omega)) \cap L_p(0,T;H_q^1(\Omega)), \\
\mathbb{E}_\tau^w(T,\Omega) &:= L_\infty(0,T;L_q(\Omega)).
\end{aligned}
$$

We also introduce the corresponding data spaces

$$
\begin{aligned}
\mathbb{F}_f^{p,q}(T,\Omega) &= L_p(0,T;L_q(\Omega)), \\
\mathbb{G}^w(T,\Omega) &:= L_\infty(0,T;L_q(\Omega)).
\end{aligned}
$$

The space $\mathbb{E}_u^{p,q,w}(T,\Omega)$ is already defined in the preliminaries and the space $\mathbb{F}_f^{p,q}(T,\Omega)$ is defined above. Let $(\tilde{w}_j,\tilde{\tau}_j)\in\mathcal{K}^{\#}(T,R_1,R_2)$, $j=1,2$, and $(w_j,\tau_j)=\Phi(\tilde{w}_j,\tilde{\tau}_j)$ be two solutions of the linearized problem. We already proved that $(w_j,\tau_j)\in\mathcal{K}^{\#}(T,R_1,R_2)$. Then, the difference of the velocity fields and the elastic parts of the stresses

$$(w_{12},\tau_{12}):=(w_2-w_1,\tau_2-\tau_1)=\Phi(\tilde{w}_2,\tilde{\tau}_2)-\Phi(\tilde{w}_1,\tilde{\tau}_1)$$

and the difference of the pressures $\pi_{12}:=\pi_2-\pi_1$ fulfill the equation

(2.32)
$$\begin{cases} \rho\partial_t w_{12}-\alpha\Delta w_{12}+\nabla\pi_{12} &= \operatorname{Div}(F_w(\tilde{w}_2)-F_w(\tilde{w}_1)+F_\tau(\tilde{\tau}_2)-F_\tau(\tilde{\tau}_1)) & \text{in } (0,T)\times\Omega,\\ \operatorname{div} u &= 0 & \text{in } (0,T)\times\Omega,\\ \partial_t\tau_{12}+(\tilde{w}_1+u_*).\nabla\tau_{12} &= G(\tilde{w}_2,\tilde{\tau}_2)-G(\tilde{w}_1,\tilde{\tau}_1)-(\tilde{w}_2-\tilde{w}_1).\nabla\tau_2 & \text{in } (0,T)\times\Omega,\\ u_{12} &= 0 & \text{on } (0,T)\times\partial\Omega,\\ u_{12}(0) &= 0 & \text{in } \Omega,\\ \tau_{12}(0) &= 0 & \text{in } \Omega. \end{cases}$$

The nonlinearities on the right-hand side of equation (2.32) are the next subject. In the case of an Oldroyd-B fluids, the following lemma holds.

Lemma 2.13. *Fix $n\in\mathbb{N}$, $n\geq 2$, $T_0,R_0,R_2>0$, $1<p<\infty$, $p\neq 2$, and $n<q<\infty$. Let $\Omega\subset\mathbb{R}^n$ be a domain with a uniform C^2-boundary. Assume that G has the special form*

$$G(w,\tau)=-\beta\tau+\gamma E(w+u_*)+\delta((\nabla(w+u_*))^T\tau+\tau\nabla(w+u_*)),$$

with $\beta,\gamma,\delta\in\mathbb{R}$, and that $\mu\in\mathbb{R}$ is constant. Then, there exists a constant $C>0$ and a function $O\colon\mathbb{R}_+\to\mathbb{R}_+$ with $O(t)\to 0$ for $t\to 0$, such that for all $R_1\in(0,R_0)$, $T\in(0,T_0)$, and $(w_j,\tau_j),(\tilde{w}_j,\tilde{\tau}_j)\in\mathcal{K}^{\#}(T,R_1,R_2)$, $j=1,2$, the estimates

$$\begin{aligned} \|F_w^D(\tilde{w}_2)-F_w^D(\tilde{w}_1)\|_{\mathbb{F}_f^{p,q}(T,\Omega)} &\leq (CR_1+O(T))\|\tilde{w}_2-\tilde{w}_1\|_{\mathbb{E}_u^{p,q,w}(T,\Omega)},\\ \|F_\tau^D(\tilde{\tau}_2)-F_\tau^D(\tilde{\tau}_1)\|_{\mathbb{F}_f^{p,q}(T,\Omega)} &\leq O(T)\|\tilde{\tau}_2-\tilde{\tau}_1\|_{\mathbb{E}_\tau^{w}(T,\Omega)},\\ \|G(\tilde{w}_2,\tilde{\tau}_2)-G(\tilde{w}_1,\tilde{\tau}_1)\|_{\mathbb{G}^w(T,\Omega)} &\leq O(T)(\|\tilde{w}_2-\tilde{w}_1\|_{\mathbb{E}_u^{p,q,w}(T,\Omega)}+\|\tilde{\tau}_2-\tilde{\tau}_1\|_{\mathbb{E}_\tau^{w}(T,\Omega)}),\\ \|(\tilde{w}_2-\tilde{w}_1).\nabla\tau_2\|_{\mathbb{G}^w(T,\Omega)} &\leq O(T)\|\tilde{w}_2-\tilde{w}_1\|_{\mathbb{E}_u^{p,q,w}(T,\Omega)} \end{aligned}$$

hold.

Proof. Let $0<R_0,R_2,T_0$, $0<R_1<R_0$, $0<T<T_0$, and $(w_j,\tau_j)\in\mathcal{K}(T,R_1,R_2)$, $j=1,2$. We denote by C a generic constant and by $O\colon\mathbb{R}_+\to\mathbb{R}$ a generic function, with $O(t)\to 0$ for $t\to 0$, which may change from line to line, but is always independent of T, R_1, w_j, and τ_j, $j=1,2$.

We recall the definition $F_w^D(w)=-\rho u_*\otimes w-\rho w\otimes u_*-\rho w\otimes w$, and the embeddings (see Proposition 1.14)

$${}_0\mathbb{E}_u^{p,q}(T,\Omega)\hookrightarrow L_{3p}(0,T;L_{3q}(\Omega))\cap L_{\frac{3p}{2}}(0,T;H^1_{\frac{3q}{2}}(\Omega)) \quad\text{and}\quad {}_0\mathbb{E}_u^{p,q,w}(T,\Omega)\hookrightarrow L_{\frac{3p}{2}}(0,T;L_{\frac{3q}{2}}(\Omega))$$

with embedding constants independent of T, $0 < T < T_0$. We obtain the estimate

$$\begin{aligned}
&\|F^D_w(\tilde w_2) - F^D_w(\tilde w_1)\|_{T,\Omega,p,q} \\
&\quad \le \rho\|u_* \otimes (\tilde w_2 - \tilde w_1)\|_{T,\Omega,p,q} + \rho\|(\tilde w_2 - \tilde w_1) \otimes u_*\|_{T,\Omega,p,q} + \rho\|\tilde w_1 \otimes (\tilde w_2 - \tilde w_1)\|_{T,\Omega,p,q} \\
&\quad\quad + \rho\|\tilde w_2 \otimes (\tilde w_2 - \tilde w_1)\|_{T,\Omega,p,q} \\
&\quad \le C\big(\|u_*\|_{T,\Omega,3p,3q}\|\tilde w_2 - \tilde w_1\|_{T,\Omega,\frac{3p}{2},\frac{3q}{2}} + (\|w_2\|_{T,\Omega,3p,3q} + \|w_1\|_{T,\Omega,3p,3q})\|\tilde w_2 - \tilde w_1\|_{T,\Omega,\frac{3p}{2},\frac{3q}{2}}\big) \\
&\quad \le C\big(\|u_*\|_{T,\Omega,3p,3q} + \|w_1\|_{{}_0\mathbb{E}^{p,q}_u(T,\Omega)} + \|w_2\|_{{}_0\mathbb{E}^{p,q}_u(T,\Omega)}\big)\|\tilde w_2 - \tilde w_1\|_{\mathbb{E}^{p,q,w}_u(T,\Omega)} \\
&\quad \le (CR_1 + O(T))\|\tilde w_2 - \tilde w_1\|_{\mathbb{E}^{p,q,w}_u(T,\Omega)}.
\end{aligned}$$

Since $F^D_\tau = \mu\tau$ where $\mu \in \mathbb{R}$ is constant, we have

$$\|F^D_\tau(\tilde\tau_2) - F^D_\tau(\tilde\tau_1)\|_{T,\Omega,p,q} = |\mu|\|\tilde\tau_2 - \tilde\tau_1\|_{T,\Omega,p,q} \le |\mu|T^{\frac{1}{p}}\|\tilde\tau_2 - \tilde\tau_1\|_{T,\Omega,\infty,q} \le O(T)\|\tilde\tau_2 - \tilde\tau_1\|_{\mathbb{E}^w_\tau(T,\Omega)}.$$

The right-hand side of the transport equation is defined by

$$G(w,\tau) = -\beta\tau + \gamma E(w + u_*) + \delta((\nabla(w + u_*))^T\tau + \tau\nabla(w + u_*)).$$

The difference can be estimated in the following way:

$$\begin{aligned}
&\|G(\tilde w_2, \tilde\tau_2) - G(\tilde w_1, \tilde\tau_1)\|_{T,\Omega,1,q} \\
&\quad \le |\beta|\|\tilde\tau_2 - \tilde\tau_1\|_{T,\Omega,1,q} + |\gamma|\|\nabla(\tilde w_2 - \tilde w_1)\|_{T,\Omega,1,q} \\
&\quad\quad + 2|\delta|(\|\nabla(\tilde w_2 - \tilde w_1)\|_{T,\Omega,1,q}\|\tilde\tau_2\|_{T,\Omega,\infty,\infty} + \|\nabla(\tilde w_1 + u_*)\|_{T,\Omega,1,\infty}\|\tilde\tau_2 - \tilde\tau_1\|_{T,\Omega,\infty,q}) \\
&\quad \le C\big(T\|\tilde\tau_2 - \tilde\tau_1\|_{\mathbb{E}^w_\tau(T,\Omega)} + T^{1-\frac{1}{p}}\|\tilde w_2 - \tilde w_1\|_{\mathbb{E}^{p,q,w}_u(T,\Omega)} \\
&\quad\quad + T^{1-\frac{1}{p}}\|\tilde w_2 - \tilde w_1\|_{\mathbb{E}^{p,q,w}_u(T,\Omega)}\|\tilde\tau_2\|_{\mathbb{E}^\#_\tau(T,\Omega)} + T^{1-\frac{1}{p}}\|\tilde w_1 + u_*\|_{\mathbb{E}^{p,q}_u(T,\Omega)}\|\tilde\tau_2 - \tilde\tau_1\|_{\mathbb{E}^w_\tau(T,\Omega)}\big) \\
&\quad \le O(T)\big(\|\tilde w_2 - \tilde w_1\|_{\mathbb{E}^{p,q,w}_u(T,\Omega)} + \|\tilde\tau_2 - \tilde\tau_1\|_{\mathbb{E}^w_\tau(T,\Omega)}\big).
\end{aligned}$$

It remains to estimate the term $\|(\tilde w_2 - \tilde w_1).\nabla\tau_2\|_{\mathbb{G}^w(T,\Omega)}$:

$$\|(\tilde w_2 - \tilde w_1).\nabla\tau_2\|_{T,\Omega,1,q} \le T^{1-\frac{1}{p}}\|\tilde w_2 - \tilde w_1\|_{T,\Omega,p,\infty}\|\nabla\tau_2\|_{T,\Omega,\infty,q} \le O(T)\|\tilde w_2 - \tilde w_1\|_{\mathbb{E}^{p,q,w}_u(T,\Omega)}.$$

□

In the case of a generalized viscoelastic fluid, we have the following lemma.

Lemma 2.14. *Fix $n \in \mathbb{N}$, $n \ge 2$, $T_0, R_0, R_2 > 0$, and $1 < p, q < \infty$ with $\frac{1}{p} + \frac{n}{2q} < \frac{1}{2}$. Let $\Omega \subset \mathbb{R}^n$ be a domain with a uniform C^2-boundary. Assume that $G(w,\tau) = g(\nabla(w + u_*), \tau)$,*

$$g \in C^1(\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}, \mathbb{R}^{n\times n}) \quad \text{with} \quad g(0,0) = 0 \quad \text{and} \quad \mu \in C^1(\mathbb{R}^{n\times n}, \mathbb{R}^{n\times n}).$$

Then, there exists a constant $C > 0$ and a function $O\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $O(t) \to 0$ for $t \to 0$ such that for all $R_1 \in (0, R_0)$, $T \in (0, T_0)$, and $(w_j, \tau_j), (\tilde w_j, \tilde\tau_j) \in \mathcal{K}^\#(T, R_1, R_2)$, $j = 1, 2$, the estimates

$$\begin{array}{lcl}
\|F^D_w(\tilde w_2) - F^D_w(\tilde w_1)\|_{\mathbb{F}^{p,q}_f(T,\Omega)} & \le & (CR_1 + O(T))\|\tilde w_2 - \tilde w_1\|_{\mathbb{E}^{p,q,w}_u(T,\Omega)}, \\
\|F^D_\tau(\tilde\tau_2) - F^D_\tau(\tilde\tau_1)\|_{\mathbb{F}^{p,q}_f(T,\Omega)} & \le & O(T)\|\tilde\tau_2 - \tilde\tau_1\|_{\mathbb{E}^w_\tau(T,\Omega)}, \\
\|G(\tilde w_2, \tilde\tau_2) - G(\tilde w_1, \tilde\tau_1)\|_{\mathbb{G}^w(T,\Omega)} & \le & O(T)(\|\tilde w_2 - \tilde w_1\|_{\mathbb{E}^{p,q,w}_u(T,\Omega)} + \|\tilde\tau_2 - \tilde\tau_1\|_{\mathbb{E}^w_\tau(T,\Omega)}), \\
\|(\tilde w_2 - \tilde w_1).\nabla\tau_2\|_{\mathbb{G}^w(T,\Omega)} & \le & O(T)\|\tilde w_2 - \tilde w_1\|_{\mathbb{E}^{p,q,w}_u(T,\Omega)}
\end{array}$$

hold.

Proof. The estimates of $F^D_w(\tilde w_2) - F^D_w(\tilde w_1)$ and $(\tilde w_2 - \tilde w_1).\nabla\tau_2$ are proved in a more general setting in the previous lemma. Here, we only prove the estimates of

$$F_\tau(\tilde\tau_2) - F_\tau(\tilde\tau_1) \quad \text{and} \quad G(\tilde w_2, \tilde\tau_2) - G(\tilde w_1, \tilde\tau_1).$$

Let $0 < R_0, R_2, T_0$, $0 < R_1 < R_0$, $0 < T < T_0$, and $(w_j, \tau_j) \in \mathcal{K}(T, R_1, R_2)$, $j = 1, 2$. We denote by C a generic constant and by $O\colon \mathbb{R}_+ \to \mathbb{R}_+$ a generic function, with $O(t) \to 0$ for $t \to 0$, which may change from line to line, but is always independent of T, R_1, w_j, and τ_j, $j = 1, 2$.

By the proposition on embedding theorems (Proposition 1.14), we have

$$\|w_j\|_{L_\infty(0,T;W^1_\infty(\Omega))\cap L_\infty(0,T;H^1_q(\Omega))} + \|u_*\|_{L_\infty(0,T;W^1_\infty(\Omega))\cap L_\infty(0,T;H^1_q(\Omega))} + \|\tau_j\|_{L_\infty(0,T;L_\infty(\Omega))} \le C_*, \qquad j = 1, 2.$$

The mean value theorem yields

$$\begin{aligned} \|F^D_\tau(\tilde\tau_2) - F^D_\tau(\tilde\tau_1)\|_{T,\Omega,p,q} &= \|\mu(\tilde\tau_2) - \mu(\tilde\tau_1)\|_{T,\Omega,p,q} \\ &\le \sup_{|\eta|<C_*} |(\nabla\mu)(\eta)| \|\tilde\tau_2 - \tilde\tau_1\|_{T,\Omega,p,q} \\ &\le \sup_{|\eta|<C_*} |(\nabla\mu)(\eta)| T^{\frac{1}{p}} \|\tilde\tau_2 - \tilde\tau_1\|_{T,\Omega,\infty,q} \\ &\le O(T) \|\tilde\tau_2 - \tilde\tau_1\|_{\mathbb{E}^{p,q,w}_\tau(T,\Omega)}, \end{aligned}$$

and

$$\begin{aligned} &\|G(\tilde w_2, \tilde\tau_2) - G(\tilde w_1, \tilde\tau_1)\|_{T,\Omega,1,q} \\ &\quad = \|g(\nabla(\tilde w_2 + u_*), \tilde\tau_2) - g(\nabla(\tilde w_1 + u_*), \tilde\tau_1)\|_{T,\Omega,1,q} \\ &\quad \le \sup_{|\eta_1|,|\eta_2|<C_*} |(\nabla g)(\eta_1,\eta_2)| \big(\|\nabla(\tilde w_2 - \tilde w_1)\|_{T,\Omega,1,q} + \|\tilde\tau_2 - \tilde\tau_1\|_{T,\Omega,1,q}\big) \\ &\quad \le \sup_{|\eta_1|,|\eta_2|<C_*} |(\nabla g)(\eta_1,\eta_2)| \big(T^{1-\frac{1}{p}} \|\nabla(\tilde w_2 - \tilde w_1)\|_{T,\Omega,p,q} + T\|\tilde\tau_2 - \tilde\tau_1\|_{T,\Omega,\infty,q}\big) \\ &\quad \le O(T)\big(\|\tilde w_2 - \tilde w_1\|_{\mathbb{E}^{p,q,w}_u(T,\Omega)} + \|\tilde\tau_2 - \tilde\tau_1\|_{\mathbb{E}^w_\tau(T,\Omega)}\big). \end{aligned}$$

$\square$

By the estimate on the solution of the Stokes problem with a right-hand side in divergence form (Proposition (2.10)), the proposition on the transport equation (Proposition 1.10), and the two previous lemmas (Lemma 2.13 and 2.14), it follows in the case of an Oldroyd-B fluid and in the case of a generalized viscoelastic fluid

$$\begin{aligned} &\|w_{12}\|_{\mathbb{E}^{p,q,w}_u(T,\Omega)} \\ &\quad = \|(\tilde\Phi_{0,0}(\tilde w_1, \operatorname{Div}(F^D_w(\tilde w_2) - F^D_w(\tilde w_1) + F^D_\tau(\tilde\tau_2) - F^D_\tau(\tilde\tau_1)), \\ &\qquad\qquad G(\tilde w_2, \tilde\tau_2) - G(\tilde w_1, \tilde\tau_1) + (\tilde w_2 - \tilde w_1).\nabla\tau_2))_1\|_{\mathbb{E}^{p,q,w}_u(T,\Omega)} \\ &\quad \le C\|F^D_w(\tilde w_2) - F^D_w(\tilde w_1) + F^D_\tau(\tilde\tau_1) - F^D_\tau(\tilde\tau_1)\|_{\mathbb{F}^{p,q}_f(T,\Omega)} \\ &\quad \le C\big(\|F^D_w(\tilde w_2) - F^D_w(\tilde w_1)\|_{\mathbb{F}^{p,q}_f(T,\Omega)} + \|F^D_\tau(\tilde\tau_2) - F^D_\tau(\tilde\tau_1)\|_{\mathbb{F}^{p,q}_f(T,\Omega)}\big) \\ &\quad \le (CR_1 + O(T))\big(\|\tilde w_2 - \tilde w_1\|_{\mathbb{E}^{p,q,w}_u(T,\Omega)} + \|\tilde\tau_2 - \tilde\tau_1\|_{\mathbb{E}^w_\tau(T,\Omega)}\big), \end{aligned}$$

and

$$
\begin{aligned}
&\|\tau_{12}\|_{\mathbb{E}^w_\tau(T,\Omega)} \\
&\quad = \|(\tilde{\Phi}_{0,0}(\tilde{w}_1, \operatorname{Div}(F^D_w(\tilde{w}_2) - F^D_w(\tilde{w}_1) + F^D_\tau(\tilde{\tau}_2) - F^D_\tau(\tilde{\tau}_1)), \\
&\qquad\qquad G(\tilde{w}_2, \tilde{\tau}_2) - G(\tilde{w}_1, \tilde{\tau}_1) + (\tilde{w}_2 - \tilde{w}_1).\nabla\tau_2))_2\|_{\mathbb{E}^w_\tau(T,\Omega)} \\
&\quad \le \|G(\tilde{w}_2, \tilde{\tau}_2) - G(\tilde{w}_1, \tilde{\tau}_1) + (\tilde{w}_2 - \tilde{w}_1).\nabla\tau_2\|_{\mathbb{G}^w(T,\Omega)} \\
&\quad \le \|G(\tilde{w}_2, \tilde{\tau}_2) - G(\tilde{w}_1, \tilde{\tau}_1)\|_{\mathbb{G}^w(T,\Omega)} + \|(\tilde{w}_2 - \tilde{w}_1).\nabla\tau_2\|_{\mathbb{G}^w(T,\Omega)} \\
&\quad \le O(T)\big(\|\tilde{w}_2 - \tilde{w}_1\|_{{}_0\mathbb{E}^{p,q}_u(T,\Omega)} + \|\tilde{\tau}_2 - \tilde{\tau}_1\|_{\mathbb{E}^w_\tau(T,\Omega)}\big).
\end{aligned}
$$

Choosing $0 < R_1, T$ sufficiently small the contraction property in $\mathbb{E}^w(T,\Omega)$ follows, more precisely

$$
\begin{aligned}
\|\Phi(\tilde{w}_2, \tilde{w}_2) - \Phi(\tilde{w}_1, \tilde{w}_1)\|_{\mathbb{E}^w(T,\Omega)} &= \|w_{12}\|_{\mathbb{E}^{p,q,w}_u(T,\Omega)} + \|\tau_{12}\|_{\mathbb{E}^w_\tau(T,\Omega)} \\
&\le \frac{1}{2}\big(\|\tilde{w}_2 - \tilde{w}_1\|_{\mathbb{E}^{p,q,w}_u(T,\Omega)} + \|\tilde{\tau}_2 - \tilde{\tau}_1\|_{\mathbb{E}^w_\tau(T,\Omega)}\big).
\end{aligned}
$$

Application of the fixed point argument and completion of the proof

Application of Proposition 1.13 proves the existence of a unique fixed point of Φ or equivalently, a unique solution

$$(u,\tau) \in H^1_p(0,T; L_{q,\sigma}(\Omega)) \cap L_p(0,T; D(A_q)) \times L_\infty(0,T; H^1_q(\Omega))$$

of (2.19). The corresponding pressure is defined by

$$\nabla\pi = \big(1 - P_q\big)\big(\operatorname{Div}\mu(\tau) - f - \rho(\partial_t u + u.\nabla u) + \alpha\Delta u\big) \in L_p(0,T; L_q(\Omega)).$$

It remains to prove that τ admits actually more time regularity. The function τ in particular fulfills the transport equation

$$\partial_t\tau + (w + u_*).\nabla\tau = G(w,\tau).$$

In Lemma 2.11, we proved in the Oldroyd-B fluid case that $G(w,\tau) \in L_p(0,T; L_q(\Omega))$ and in Lemma 2.12 in the general viscoelastic fluid case that $G(w,\tau) \in L_\infty(0,T; L_q(\Omega))$. This implies $\partial_t\tau \in L_p(0,T; L_q(\Omega))$ in the case of an Oldroyd-B fluid and $\partial_t\tau \in L_\infty(0,T; L_q(\Omega))$ in the general case, by proposition on the transport equation (Proposition 1.10). This completes the proof. □

2.3 Generalized viscoelastic fluids on the half space with perfect slip boundary conditions

In Section 2.1, we analyse (2.1) in the case of a bounded domain with Dirichlet and perfect slip boundary conditions and in Section (2.2), we investigate (2.1) in unbounded domains with Dirichlet boundary conditions. This section is intended to consider (2.1) in an unbounded domain with perfect slip boundary conditions. We will restrict our attention to the analysis of (2.1) in the case

$\alpha > 0$ is constant, $g(0,0) = \mu(0) = 0$, $\Omega = \mathbb{R}^n_+$, and $\Gamma_D = \emptyset$, i.e. we consider

$$(2.33)\qquad \left\{\begin{array}{rcll} \rho(\partial_t u + u.\nabla u) - \alpha\Delta u + \nabla\pi &=& \operatorname{Div}\mu(\tau) + f & \text{in } (0,T_0)\times\mathbb{R}^n_+, \\ \operatorname{div} u &=& 0 & \text{in } (0,T_0)\times\mathbb{R}^n_+, \\ \partial_t\tau + u.\nabla\tau &=& g(\nabla u,\tau) & \text{in } (0,T_0)\times\mathbb{R}^n_+, \\ -(u_n,(\alpha\partial_n u_j + \mu(\tau)_{j,n})_{j=1,\dots,n-1}) &=& 0 & \text{on } (0,T_0)\times\partial\mathbb{R}^n_+, \\ u(0) &=& u_0 & \text{in } \mathbb{R}^n_+, \\ \tau(0) &=& \tau_0 & \text{in } \mathbb{R}^n_+. \end{array}\right.$$

We emphasise that in this situation (2.33) is consistent with (2.1), since $\nu = -e_n$ and thus

$$-(u\cdot\nu,[2\alpha Eu\nu + \mu(\tau)\nu]_{\tan}) = (u_n,(\alpha\partial_n u_j + \mu(\tau)_{j,n})_{j=1}^{n-1}) = 0 \quad \text{on } (0,T_0)\times\partial\mathbb{R}^n_+.$$

The compatibility conditions (2.2) take the form

$$(2.34)\qquad \operatorname{div} u_0 = 0 \quad \text{in } \mathbb{R}^n_+ \quad \text{and} \quad -(u_{0,n},(\alpha\partial_n u_{0,j} + \mu(\tau_0)_{j,n})_{j=1}^{n-1}) = 0 \quad \text{on } \partial\mathbb{R}^n_+.$$

We prove the existence of a small time interval $(0,T)$ and the existence of a unique strong solution on this time interval in the regularity class

$$\begin{aligned} & u \in H^1_p(0,T;L_p(\mathbb{R}^n_+)) \cap L_p(0,T;H^2_p(\mathbb{R}^n_+)), \quad \pi \in L_p(0,T;\widehat{H}^1_p(\mathbb{R}^n_+)), \\ & \text{and} \quad \tau \in W^1_\infty(0,T;L_p(\mathbb{R}^n_+)) \cap L_\infty(0,T;H^1_p(\mathbb{R}^n_+)). \end{aligned}$$

Let us state the main theorem of this section.

Theorem 2.15. *Fix $n \in \mathbb{N}$, $n \geq 2$, $n+2 < p < \infty$, and $T_0,\rho,\alpha > 0$. We assume, that*

$$\mu \in C^3(\mathbb{R}^{n\times n},\mathbb{R}^{n\times n}) \quad \text{with} \quad \mu(0) = 0 \quad \text{and} \quad g \in C^1(\mathbb{R}^{n\times n}\times\mathbb{R}^{n\times n},\mathbb{R}^{n\times n}) \quad \text{with} \quad g(0,0) = 0.$$

Then, for each $f \in L_p(0,T_0;L_p(\mathbb{R}^n_+))$, $u_0 \in W_p^{2-\frac{2}{p}}(\mathbb{R}^n_+)$, and $\tau_0 \in H^1_p(\mathbb{R}^n_+)$, satisfying the compatibility conditions

$$\operatorname{div} u_0 = 0 \quad \text{in } \mathbb{R}^n_+ \quad \text{and} \quad (u_{0,n},(\alpha\partial_n u_{0,j} + \mu(\tau_0)_{j,n})_{j=1}^{n-1}) = 0 \quad \text{on } \partial\mathbb{R}^n_+,$$

there exists a time $0 < T < T_0$ and unique strong solution of (2.1) on the time interval $(0,T)$ in the regularity class

$$\begin{aligned} & u \in H^1_p(0,T;L_p(\mathbb{R}^n_+)) \cap L_p(0,T;H^2_p(\mathbb{R}^n_+)), \quad \pi \in L_p(0,T;\widehat{H}^1_p(\mathbb{R}^n_+)), \\ & \textit{and} \quad \tau \in W^1_\infty(0,T;L_p(\mathbb{R}^n_+)) \cap L_\infty(0,T;H^1_p(\mathbb{R}^n_+)). \end{aligned}$$

Sketch of the proof

The ideas of the proof are similar to the ideas used in the previous section. We apply a modified version of the contraction mapping principle (Proposition 1.13). The main difference to the previous section is the prove of the estimate of the velocity field in

$$H_p^{\frac{1}{2}}(0,T;L_p(\mathbb{R}^n_+)) \cap L_p(0,T;H^1_p(\mathbb{R}^n_+))).$$

In the previous section, the proof of this estimate is based on the bounded imaginary powers of the Stokes operator. This method is, in the case of perfect slip boundary condition, not directly applicable, since the linearization represents a Stokes problem with inhomogeneous boundary data (see (2.6) and (2.40)). Instead, we use an explicit representation formula of the solution of the Stokes equation in the half space in order to prove the required estimate.

2.3.1 An L_p-estimate for the Stokes problem with inhomogeneous perfect slip boundary conditions in the half space

The aim of this subsection is to prove, that the solution $u \in H_p^1(0,T;L_p(\mathbb{R}_+^n)) \cap L_p(0,T;H_p^2(\mathbb{R}_+^n))$ of the Stokes system with inhomogeneous perfect slip boundary condition

$$
(2.35) \qquad \begin{cases} \rho\partial_t u - \alpha\Delta u + \nabla\pi &= \operatorname{Div} F & \text{in } (0,T)\times\mathbb{R}_+^n, \\ \operatorname{div} u &= 0 & \text{in } (0,T)\times\mathbb{R}_+^n, \\ -(u_n, \alpha(\partial_n u_j)_{j=1,\dots,n-1}) &= (0,(F_{j,n})_{j=1,\dots,n-1}) & \text{on } (0,T)\times\partial\mathbb{R}_+^n, \\ u(0) &= 0 & \text{in } \mathbb{R}_+^n, \end{cases}
$$

with zero initial value and right-hand side $\operatorname{Div} F$, where

$$
F \in L_p(0,T;H_p^1(\mathbb{R}_+^n)) \quad \text{and} \quad \gamma_{\partial\mathbb{R}_+^n} F \in W_p^{\frac{1}{2p}}(0,T;L_p(\partial\mathbb{R}_+^n))
$$

satisfies the compatibility condition

$$
F_{j,n}(0) = 0 \quad \text{on } \partial\mathbb{R}_+^n, \quad j = 1,\dots,n-1,
$$

can be estimated by

$$
\|u\|_{H_p^{\frac{1}{2}}(0,T;L_p(\mathbb{R}_+^n))\cap L_p(0,T;H_p^1(\mathbb{R}_+^n))} \le C\|F\|_{T,\mathbb{R}_+^n,p,p}.
$$

The proof is based on an explicit solution formula of (2.35) and explicit commutator relations of the resolvent of the Dirichlet and Neumann Laplace operator with the normal derivative.

The following proposition is analogous to Proposition 2.10.

Proposition 2.16. *Fix $n \in \mathbb{N}$, $n \ge 2$, $1 < p < \infty$ with $p \ne 2$, and $T_0, \rho, \alpha > 0$. Then, there exists a constant $C > 0$ such that for each $0 < T < T_0$ and right-hand side* $\operatorname{Div} F$*, with $F \in L_p(0,T;H_p^1(\mathbb{R}_+^n))$ and $\gamma_{\partial\mathbb{R}_+^n} F \in W_p^{\frac{1}{2p}}(0,T;L_p(\partial\mathbb{R}_+^n))$, satisfying the compatibility condition*

$$
F_{j,n}(0) = 0 \quad \text{on } \partial\mathbb{R}^n, \quad j = 1,\dots,n-1,
$$

the unique solution $u \in H_p^1(0,T;L_p(\mathbb{R}_+^n)) \cap L_p(0,T;H_p^2(\mathbb{R}_+^n))$ of (2.35) can be estimated by

$$
\|u\|_{{}_0H_p^{\frac{1}{2}}(0,T;L_p(\mathbb{R}_+^n))\cap L_p(0,T;H_p^1(\mathbb{R}_+^n))} \le C\|F\|_{T,\mathbb{R}_+^n,p,p}.
$$

Proof. There is no loss of generality in assuming $\alpha = \rho = 1$. With C, we always denote a generic constant, which may change from line to line, but is always independent of F and T, $0 < T < T_0$. We denote by

$$
\Delta' \colon L_p(0,T;L_p(\mathbb{R}^{n-1})) \to L_p(0,T;L_p(\mathbb{R}^{n-1})), \quad D(\Delta') = L_p(0,T;H_p^2(\mathbb{R}^{n-1})),
$$

the Laplace operator on the tangential components $\mathbb{R}^{n-1}$, and by

$$
\partial_t \colon L_p(0,T;L_p(\mathbb{R}^{n-1})) \to L_p(0,T;L_p(\mathbb{R}^{n-1})), \quad D(\partial_t) = {}_0H_p^1(0,T;L_p(\mathbb{R}^{n-1}))
$$

the time derivative. By Kalton and Weis [KW01] and Denk, Saal and Seiler [DSS08], the two operators Δ' and ∂_t admit a joint $\mathcal{H}^\infty$-calculus. Further, we denote by Δ_D the Dirichlet-Laplace operator on $\mathbb{R}^n_+$, i.e. the operator

$$\Delta_D\colon L_p(\mathbb{R}^n_+) \to L_p(\mathbb{R}^n_+), \quad D(\Delta_D) = H^2_p(\mathbb{R}^n_+) \cap H^1_{p,0}(\mathbb{R}^n_+),$$

and by Δ_N the Neumann-Laplace operator on $\mathbb{R}^n_+$, i.e. the operator

$$\Delta_N\colon L_p(\mathbb{R}^n_+) \to L_p(\mathbb{R}^n_+), \quad D(\Delta_N) = \{u \in H^2_p(\mathbb{R}^n_+)\colon \partial_n u = 0 \text{ on } \partial\mathbb{R}^n_+\}.$$

The solution operator to the Laplace problem with Dirichlet boundary data and zero initial value

$$u' - \Delta_D u = f \quad \text{in } (0,T), \qquad u(0) = 0$$

is denoted by $u = (\partial_t - \Delta_D)^{-1} f$ and the corresponding solution operator with Neumann boundary conditions

$$u' - \Delta_N u = f \quad \text{in } (0,T), \qquad u(0) = 0$$

is denoted by $(\partial_t - \Delta_N)^{-1} f$.

Due to Proposition 1.8, there exists a unique solution

$$(u,\pi) \in {}_0H^1_p(0,T;L_p(\mathbb{R}^n_+)) \cap L_p(0,T;H^2_p(\mathbb{R}^n_+)) \times L_p(0,T;\widehat{H}^1_p(\mathbb{R}^n_+))$$

of (2.35). To estimate the solution, we give an explicit representation formula. We define

$$v_j(x_n) := (\partial_t - \Delta_N)^{-1}((\operatorname{Div} F)_j - \partial_j \pi)(x_n) + \sqrt{\partial_t - \Delta'}^{-1} e^{-\sqrt{\partial_t - \Delta'}x_n} \gamma_{\partial\mathbb{R}^n_+} F_{j,n}, \qquad x_n > 0, \quad j = 1,\dots,n-1,$$

and

$$v_n := (\partial_t - \Delta_D)^{-1}((\operatorname{Div} F)_n - \partial_n \pi),$$

where the operator

$$\sqrt{\partial_t - \Delta'}^{-1} e^{-\sqrt{\partial_t - \Delta'}x_n}, \quad x_n > 0$$

is defined via the joint $\mathcal{H}^\infty$-calculus of ∂_t and Δ'. Substituting v into (2.35), we see that v solves (2.35), and hence $u = v$. Next, we investigate this solution formula. We use the known representation of the commutators $[\partial_n, (\partial_t - \Delta_D)^{-1}]$ and $[\partial_n, (\partial_t - \Delta_N)^{-1}]$, i.e. for $\tilde f \in L_p(0,T;H^1_q(\mathbb{R}^n_+))$, it holds

$$\begin{aligned}(\partial_n(\partial_t - \Delta_D)^{-1}\tilde f)(x_n) &= ((\partial_t - \Delta_N)^{-1}\partial_n \tilde f)(x_n) + \sqrt{\partial_t - \Delta'}^{-1} e^{-\sqrt{\partial_t - \Delta'}x_n} \gamma_{\partial\mathbb{R}^n_+}\tilde f, \quad x_n > 0,\\ \partial_n(\partial_t - \Delta_N)^{-1}\tilde f &= (\partial_t - \Delta_D)^{-1}\partial_n \tilde f.\end{aligned}$$

Applying the last two equations yields

(2.36)

$$\begin{aligned} u_j(x_n) &= (\partial_t - \Delta_N)^{-1}((\operatorname{Div} F)_j - \partial_j\pi)(x_n) + \sqrt{\partial_t - \Delta'}^{-1} e^{-\sqrt{\partial_t - \Delta'}x_n} \gamma_{\partial\mathbb{R}^n_+} F_{j,n}\\ &= \sum_{k=1}^{n}((\partial_t - \Delta_N)^{-1}\partial_k F_{j,k})(x_n) - ((\partial_t - \Delta_N)^{-1}\partial_j\pi)(x_n) + \sqrt{\partial_t - \Delta'}^{-1} e^{-\sqrt{\partial_t - \Delta'}x_n} \gamma_{\partial\mathbb{R}^n_+} F_{j,n}\\ &= \sum_{k=1}^{n-1}(\partial_k(\partial_t - \Delta_N)^{-1}F_{j,k})(x_n) + (\partial_n(\partial_t - \Delta_D)^{-1}F_{j,n})(x_n) - ((\partial_t - \Delta_N)^{-1}\partial_j\pi)(x_n), \end{aligned}$$

where $x_n > 0$ and $j = 1, \dots, n-1$, as well as

$$
\begin{aligned}
u_n &= (\partial_t - \Delta_D)^{-1} (\operatorname{Div} F)_n - (\partial_t - \Delta_D)^{-1} \partial_n \pi \\
&= \sum_{k=1}^{n} (\partial_t - \Delta_D)^{-1} \partial_k F_{n,k} - (\partial_t - \Delta_D)^{-1} \partial_n \pi \\
&= \sum_{k=1}^{n-1} \partial_k (\partial_t - \Delta_D)^{-1} F_{n,k} + \partial_n (\partial_t - \Delta_N)^{-1} F_{n,n} - (\partial_t - \Delta_D)^{-1} \partial_n \pi.
\end{aligned} \tag{2.37}
$$

Next, we show that the operator

$$
(\partial_t - \Delta_{\mathcal{B}})^{-1} := \begin{pmatrix} (\partial_t - \Delta_N)^{-1} \\ \vdots \\ (\partial_t - \Delta_N)^{-1} \\ (\partial_t - \Delta_D)^{-1} \end{pmatrix}
$$

commutes with the Helmholtz projection. Saal [Saa06, combining (27) and (28)] proved, that

$$
(\partial_t - \Delta_{\mathcal{B}})^{-1} \colon L_p(0,T; L_{p,\sigma}(\mathbb{R}^n_+)) \to L_p(0,T; L_{p,\sigma}(\mathbb{R}^n_+)). \tag{2.38}
$$

Our aim is now to show that

$$
(\partial_t - \Delta_{\mathcal{B}})^{-1} \colon L_p(0,T; G_p(\mathbb{R}^n_+)) \to L_p(0,T; G_p(\mathbb{R}^n_+)). \tag{2.39}
$$

Let $\theta \in L_p(0,T; \widehat{H}^1_p(\mathbb{R}^n))$ and we prove $((\partial_t - \Delta_{\mathcal{B}})^{-1} \nabla\theta | g)_{T,\mathbb{R}^n_+}$ for $g \in L_{p'}(0,T; L_{p'}(\mathbb{R}^n_+))$, with $1 < p' < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, which implies (2.39). Let $g \in L_{p'}(0,T; L_{p',\sigma}(\mathbb{R}^n_+))$ and let f be the solution of

$$
\partial_t f + \Delta_{\mathcal{B}} f = g \quad \text{in } (0,T), \quad f(T) = 0.
$$

By integration by parts, it follows that

$$
\begin{aligned}
&((\partial_t - \Delta_{\mathcal{B}})^{-1} \nabla\theta | g)_{T,\mathbb{R}^n_+} \\
&\quad = ((\partial_t - \Delta_N)^{-1} \nabla'\theta | g')_{T,\mathbb{R}^n_+} + ((\partial_t - \Delta_D)^{-1} \partial_n\theta | g_n)_{T,\mathbb{R}^n_+} \\
&\quad = ((\partial_t - \Delta_N)^{-1} \nabla'\theta | \partial_t f' + \Delta f')_{T,\mathbb{R}^n_+} + ((\partial_t - \Delta_D)^{-1} \partial_n\theta | \partial_t f_n + \Delta f_n)_{T,\mathbb{R}^n_+} \\
&\quad = ((\partial_t - \Delta_N)^{-1} \nabla'\theta | \partial_t f')_{T,\mathbb{R}^n_+} + ((\partial_t - \Delta_N)^{-1} \nabla'\theta | \Delta f')_{T,\mathbb{R}^n_+} \\
&\qquad + ((\partial_t - \Delta_D)^{-1} \partial_n\theta | \partial_t f_n)_{T,\mathbb{R}^n_+} + ((\partial_t - \Delta_D)^{-1} \partial_n\theta | \Delta f_n)_{T,\mathbb{R}^n_+} \\
&\quad = -(\partial_t (\partial_t - \Delta_N)^{-1} \nabla'\theta | f')_{T,\mathbb{R}^n_+} + (\Delta (\partial_t - \Delta_N)^{-1} \nabla'\theta | f')_{T,\mathbb{R}^n_+} \\
&\qquad - ((\partial_t - \Delta_N)^{-1} \nabla'\theta | \partial_n f')_{T,\partial\mathbb{R}^n} + (\partial_n (\partial_t - \Delta_N)^{-1} \nabla'\theta | f')_{T,\partial\mathbb{R}^n_+} \\
&\qquad - (\partial_t (\partial_t - \Delta_D)^{-1} \partial_n\theta | f_n)_{T,\mathbb{R}^n_+} + (\Delta (\partial_t - \Delta_D)^{-1} \partial_n\theta | f_n)_{T,\mathbb{R}^n_+} \\
&\qquad - ((\partial_t - \Delta_D)^{-1} \partial_n\theta | \partial_n f_n)_{T,\partial\mathbb{R}^n} + (\partial_n (\partial_t - \Delta_D)^{-1} \partial_n\theta | f_n)_{T,\partial\mathbb{R}^n_+} \\
&\quad = -(\nabla\theta | f) - ((\partial_t - \Delta_N)^{-1} \nabla'\theta | \partial_n f')_{T,\partial\mathbb{R}^n} + (\partial_n (\partial_t - \Delta_N)^{-1} \nabla'\theta | f')_{T,\partial\mathbb{R}^n_+} \\
&\qquad - ((\partial_t - \Delta_D)^{-1} \partial_n\theta | \partial_n f_n)_{T,\partial\mathbb{R}^n} + (\partial_n (\partial_t - \Delta_D)^{-1} \partial_n\theta | f_n)_{T,\partial\mathbb{R}^n_+}
\end{aligned}
$$

where the boundary term in the time integration vanish due to $((\partial_t - \Delta_{\mathcal{B}})^{-1}\nabla\theta)(0) = 0$ and $f(T) = 0$. The remaining boundary integrals are also zero, since

$$\begin{aligned}
&\partial_n(\partial_t - \Delta_N)^{-1}\nabla'\theta = 0 \quad \text{on } (0,T)\times\partial\mathbb{R}^n_+, \quad \partial_n f' = 0 \quad \text{on } (0,T)\times\partial\mathbb{R}^n_+,\\
&(\partial_t - \Delta_D)^{-1}\partial_n\theta = 0 \quad \text{on } (0,T)\times\partial\mathbb{R}^n_+, \quad f_n = 0 \quad \text{on } (0,T)\times\partial\mathbb{R}^n_+.
\end{aligned}$$

Due to $(\nabla\theta|f)_{T,\mathbb{R}^n_+} = 0$, we have

$$((\partial_t - \Delta_{\mathcal{B}})^{-1}\nabla\theta|g)_{T,\mathbb{R}^n_+} = 0, \quad g \in L_{p'}(0,T;L_{p',\sigma}(\mathbb{R}^n_+)).$$

This is equivalent to (2.39). On account of (2.38) and (2.39), we can conclude that

$$\begin{aligned}
P_p(\partial_t - \Delta_{\mathcal{B}})^{-1}h &= P_p(\partial_t - \Delta_{\mathcal{B}})^{-1}P_p h + P_p(\partial_t - \Delta_{\mathcal{B}})^{-1}(1-P_p)h\\
&= (\partial_t - \Delta_{\mathcal{B}})^{-1}P_p h, \quad h \in L_p(0,T;L_p(\mathbb{R}^n_+))
\end{aligned}$$

and therefore, the operator $(\partial_t - \Delta_{\mathcal{B}})^{-1}$ commutes with the Helmholtz projection.

Since $\operatorname{div} u = 0$ in $(0,T)\times\mathbb{R}^n_+$ and $u_n = 0$ on $(0,T)\times\partial\mathbb{R}^n_+$, we obtain $P_p u = u$. Taking into account (2.36), (2.37) as well as $P_p(\partial_t - \Delta_{\mathcal{B}})^{-1}\nabla\pi = 0$, we deduce that

$$\begin{aligned}
u &= P_p u\\
&= P_p\sum_{k=1}^{n-1}\partial_k\begin{pmatrix}(\partial_t - \Delta_N)^{-1}F_{1,k}\\ \vdots\\ (\partial_t - \Delta_N)^{-1}F_{n-1,k}\\ (\partial_t - \Delta_D)^{-1}F_{n,k}\end{pmatrix} + P_p\partial_n\begin{pmatrix}(\partial_t - \Delta_D)^{-1}F_{1,n}\\ \vdots\\ (\partial_t - \Delta_D)^{-1}F_{n-1,n}\\ (\partial_t - \Delta_N)^{-1}F_{n,n}\end{pmatrix} - P_p\begin{pmatrix}(\partial_t - \Delta_N)^{-1}\partial_1\\ \vdots\\ (\partial_t - \Delta_N)^{-1}\partial_{n-1}\\ (\partial_t - \Delta_D)^{-1}\partial_n\end{pmatrix}\pi.\\
&= P_p\sum_{k=1}^{n-1}\partial_k\begin{pmatrix}(\partial_t - \Delta_N)^{-1}F_{1,k}\\ \vdots\\ (\partial_t - \Delta_N)^{-1}F_{n-1,k}\\ (\partial_t - \Delta_D)^{-1}F_{n,k}\end{pmatrix} + P_p\partial_n\begin{pmatrix}(\partial_t - \Delta_D)^{-1}F_{1,n}\\ \vdots\\ (\partial_t - \Delta_D)^{-1}F_{n-1,n}\\ (\partial_t - \Delta_N)^{-1}F_{n,n}\end{pmatrix}.
\end{aligned}$$

By the continuity of the mappings

$$\nabla(\partial_t - \Delta_D)^{-1}, \nabla(\partial_t - \Delta_N)^{-1} \colon L_p(0,T;L_p(\mathbb{R}^n_+)) \to {}_0H^{\frac{1}{2}}_p(0,T;L_p(\mathbb{R}^n_+)) \cap L_p(0,T;H^1_p(\mathbb{R}^n_+))$$

and the continuity of the Helmholtz projection in ${}_0H^{\frac{1}{2}}_p(0,T;L_p(\mathbb{R}^n_+)) \cap L_p(0,T;H^1_p(\mathbb{R}^n_+))$, it follows that

$$\|u\|_{{}_0H^{\frac{1}{2}}_p(0,T;L_p(\mathbb{R}^n_+))\cap L_p(0,T;H^1_p(\mathbb{R}^n_+))} \le C\|F\|_{T,\mathbb{R}^n_+,p,p}.$$

□

2.3.2 Proof of the main theorem

Next, we present a proof of Theorem 2.15.

Proof of Theorem 2.15. We are now in a position to follow the lines of the proof of Theorem 2.6 to prove Theorem 2.15.

Reduction to $u_0 = 0$ and $f = 0$ and fixed point formulation

We proceed the same way as in the proof of Theorem 2.1. By Runst and Sickel [RS96, Theorem 5.5.3.1], it follows that $\mu(\tau_0) \in W^{1-\frac{1}{p}}(\partial\mathbb{R}^n_+)$, since $\mu(0) = 0$. Let

$$(u_*, \pi_*) \in H^1_p(0, T_0; L_p(\mathbb{R}^n_+)) \cap L_p(0, T_0; H^2_p(\mathbb{R}^n_+)) \times L_p(0, T_0; \widehat{H}^1_p(\mathbb{R}^n_+))$$

be the solution of

$$\begin{cases} \rho\partial_t u_* - \alpha\Delta u_* + \nabla\pi_* &= f & \text{in } (0, T_0) \times \mathbb{R}^n_+, \\ \operatorname{div} u &= 0 & \text{in } (0, T_0) \times \mathbb{R}^n_+, \\ -(u_{*,n}, (\alpha\partial_n u_{*,j})_{j=1,\dots,n-1}) &= (0, (\mu(\tau_0)_{j,n})_{j=1,\dots,n}) & \text{on } (0, T_0) \times \partial\mathbb{R}^n_+, \\ u(0) &= u_0 & \text{in } \mathbb{R}^n_+, \end{cases}$$

given by Proposition 1.8. We set

$$u = w + u_* \quad \text{and} \quad \pi = \pi_* + \psi.$$

Then (u, π, τ) solves (2.33) if and only if (w, ψ, τ) solves

$$(2.40) \qquad \begin{cases} \rho\partial_t w - \alpha\Delta w + \nabla\psi &= f_* + \operatorname{Div}(F^D_w(w) + F^D_\tau(\tau)) & \text{in } (0, T_0) \times \mathbb{R}^n_+, \\ \operatorname{div} u &= 0 & \text{in } (0, T_0) \times \mathbb{R}^n_+, \\ \partial_t\tau + (w + u_*).\nabla\tau &= G(w, \tau) & \text{in } (0, T_0) \times \mathbb{R}^n_+, \\ -(w_n, \alpha(\partial_n w_j)_{j=1,\dots,n-1}) &= (0, H_\tau(\tau)) & \text{on } (0, T_0) \times \partial\mathbb{R}^n_+, \\ w(0) &= 0 & \text{in } \mathbb{R}^n_+, \\ \tau(0) &= \tau_0 & \text{in } \mathbb{R}^n_+, \end{cases}$$

where the terms f_*, F^D_w, F^D_τ, G, and H_τ are the same as in the sections above. Since α is constant and $\Omega = \mathbb{R}^n_+$ their representation simplifies. The terms on the right-hand side of the Stokes equation are given by

$$f_* = -\rho u_*.\nabla u_* + \operatorname{Div}\mu(\tau_0),$$

as well as

$$F^D_w(w) = -\rho u_* \otimes w - \rho w \otimes u_* - \rho w \otimes w \quad \text{and} \quad F^D_\tau(\tau) = \mu(\tau) - \mu(\tau_0).$$

The right-hand side of the transport equation reads

$$G(w, \tau) = g(\nabla(w + u_*), \tau).$$

The terms H_τ reduces to

$$H_\tau(\tau)_j = (\mu(\tau)_{j,n} - \mu(\tau_0)_{j,n}), \quad j = 1, \dots, n-1.$$

It is worth pointing out, that the nonlinearities have the structure required in (2.35), where we assume a relation between the right-hand side of the Stokes equation and the right-hand side of the boundary condition, i.e. for (w, τ) with $\gamma_{\partial\mathbb{R}^n_+} w_n = 0$, it holds

$$(F^D_w(w) + F^D_\tau(\tau))_{j,n} = H_\tau(\tau)_j \quad \text{on } (0, T_0) \times \partial\mathbb{R}^n_+, \quad j = 1, \dots, n-1.$$

We rewrite (2.40) in the form of a fixed point equation in a Banach space. Let $n+2<p<r<\infty$. Since the elastic part of the stress appears on the boundary, we choose the same spaces as in the proof of Theorem 2.1, i.e. the solution spaces are given by

$$\begin{array}{rcl} {}_0\mathbb{E}_{u,c}(T,\mathbb{R}^n_+) & = & \{w \in {}_0H^1_p(0,T;L_p(\mathbb{R}^n_+)) \cap L_p(0,T;H^2_p(\mathbb{R}^n_+))\colon w_n = 0 \text{ on } \partial\mathbb{R}^n_+\}, \\ \mathbb{E}_\tau(T,\mathbb{R}^n_+) & = & \widehat{H}^1_r(0,T;L_p(\mathbb{R}^n_+)) \cap L_\infty(0,T;H^1_p(\mathbb{R}^n_+)), \end{array}$$

the space for the velocity field, where we do not prescribe the initial value, by

$$\mathbb{E}_u(T,\mathbb{R}^n_+) \quad = \quad H^1_p(0,T;L_p(\mathbb{R}^n_+)) \cap L_p(0,T;H^2_p(\mathbb{R}^n_+)),$$

and the spaces for the data are given by

$$\begin{array}{rcl} \mathbb{F}_f(T,\mathbb{R}^n_+) & = & L_p(0,T;L_p(\mathbb{R}^n_+)), \\ \mathbb{G}(T,\mathbb{R}^n_+) & = & L_r(0,T;L_p(\mathbb{R}^n_+)) \cap L_1(0,T;H^1_p(\mathbb{R}^n_+)), \\ {}_0\mathbb{H}_u(T,\partial\mathbb{R}^n_+) & = & {}_0W_p^{\frac{1}{2}-\frac{1}{2p}}(0,T;L_p(\partial\mathbb{R}^n_+)) \cap L_p(0,T;W_p^{1-\frac{1}{p}}(\partial\mathbb{R}^n_+)). \end{array}$$

The fixed point map, corresponding to problem (2.40) is exactly defined as in the proof of Theorem 2.1, i.e.

(2.41)

$$\begin{array}{rcl} \Phi\colon {}_0\mathbb{E}_{u,c}(T,\mathbb{R}^n_+) \times \mathbb{E}_\tau(T,\mathbb{R}^n_+) & \to & {}_0\mathbb{E}_{u,c}(T,\mathbb{R}^n_+) \times \mathbb{E}_\tau(T,\mathbb{R}^n_+), \\ (w,\tau) & \mapsto & \tilde{\Phi}_{0,\tau_0}(w, f_* + \operatorname{Div} F^D_w(w) + \operatorname{Div} F^D_\tau(\tau), G(w,\tau), H_\tau(\tau)), \end{array}$$

where

$$\begin{array}{rcl} \tilde{\Phi}_{0,\tau_0}\colon {}_0\mathbb{E}_{u,c}(T,\mathbb{R}^n_+) \times \mathbb{F}_f(T,\mathbb{R}^n_+) \times \mathbb{G}(T,\mathbb{R}^n_+) \times {}_0\mathbb{H}_u(T,\partial\mathbb{R}^n_+) & \to & {}_0\mathbb{E}_{u,c}(T,\mathbb{R}^n_+) \times \mathbb{E}_\tau(T,\mathbb{R}^n_+), \\ (\tilde{w},\tilde{f},\tilde{g},\tilde{h}) & \mapsto & (w,\tau) \end{array}$$

denotes the solution operator to the following problem:

$$(2.42) \qquad \left\{ \begin{array}{rcll} \rho\partial_t w - \Delta w + \nabla\psi & = & \tilde{f} & \text{in } (0,T_0)\times\mathbb{R}^n_+, \\ \operatorname{div} w & = & 0 & \text{in } (0,T_0)\times\mathbb{R}^n_+, \\ \partial_t\tau + (\tilde{w}+u_*).\nabla\tau & = & \tilde{g} & \text{in } (0,T_0)\times\mathbb{R}^n_+, \\ -(w_n, \alpha(\partial_n w_j)_{j=1}^n) & = & (0,\tilde{h}) & \text{on } (0,T_0)\times\partial\mathbb{R}^n_+, \\ w(0) & = & 0 & \text{in } \mathbb{R}^n_+, \\ \tau(0) & = & \tau_0 & \text{in } \mathbb{R}^n_+. \end{array} \right.$$

Taking into account Lemma 2.17, the fact that Φ and $\tilde{\Phi}_{0,\tau_0}$ are well defined follows the same way as in the proof of Theorem 2.1 as well as in the proof of Theorem 2.6.

Analysis of Φ

Next, we analyse Φ. For $0<R_1,R_2,R_3<\infty$, $0<T<T_0$, and $n+2<p<r<\infty$, we recall the definitions

$$\begin{array}{rcl} \mathcal{K}_w(T,R_1) & = & \{w \in {}_0\mathbb{E}_{u,c}(T,\mathbb{R}^n_+)\colon \|w\|_{{}_0\mathbb{E}_u(T,\mathbb{R}^n_+)} < R_1\}, \\ \mathcal{K}_\tau(T,R_2,R_3) & = & \{\tau \in \mathbb{E}_\tau(T,\mathbb{R}^n_+)\colon \tau(0)=\tau_0,\ \|\tau\|_{L_\infty(0,T;H^1_q(\mathbb{R}^n_+))} \le R_2 \\ & & \quad \text{and } \|\partial_t\tau\|_{T,\Omega,r,p} < R_3\}, \\ \mathcal{K}(T,R_1,R_2,R_3) & = & \mathcal{K}_w(T,R_1) \times \mathcal{K}_\tau(T,R_2,R_3). \end{array}$$

The map Φ maps $\mathcal{K}(T, R_1, R_2, R_3)$ into itself

We prove, that we can choose T, R_1, R_2, and R_3, such that

$$\Phi(\mathcal{K}(T, R_1, R_2, R_3)) \subset \mathcal{K}(T, R_1, R_2, R_3).$$

For this purpose, we estimate the nonlinearities in the following lemma.

Lemma 2.17. *Fix $n \in \mathbb{N}$, $n \geq 2$, $T_0, R_0, R_2, R_3 > 0$, and $n+2 < p < r < \infty$. Assume that*

$$\mu \in C^3(\mathbb{R}^{n\times n}, \mathbb{R}^{n\times n}) \quad \text{and} \quad g \in C^1(\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}, \mathbb{R}^{n\times n}) \quad \text{with} \quad g(0,0) = 0.$$

Then, there exists a constant $C > 0$ and a function $O\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $O(t) \to 0$ for $t \to 0$, such that for all $R_1 \in (0, R_0)$, $T \in (0, T_0)$, and $(w, \tau) \in \mathcal{K}(T, R_1, R_2, R_3)$ the estimates

$$\begin{array}{lcl} \|\operatorname{Div} F_w^D(w)\|_{\mathbb{F}_f(T,\mathbb{R}^n_+)} & \leq & CR_1^2 + O(T), \\ \|\operatorname{Div} F_\tau^D(\tau)\|_{\mathbb{F}_f(T,\mathbb{R}^n_+)} & \leq & O(T), \\ \|H_\tau(\tau)\|_{{}_0\mathbb{H}_u(T,\partial\mathbb{R}^n_+)} & \leq & O(T) \\ \|G(w,\tau)\|_{\mathbb{G}(T,\mathbb{R}^n_+)} & \leq & O(T), \\ \|G(w,\tau)\|_{T,\mathbb{R}^n_+,\infty,p} & \leq & C \end{array}$$

hold.

Proof. We already proved the estimate of $\operatorname{Div} F_w^D(w)$ and $\operatorname{Div} F_\tau^D(\tau)$ in Lemma 2.12. The estimate of $H_\tau(\tau)$ is proved in Lemma 2.3 in the case that domain is bounded. The same proof also holds for the half space. Furthermore, we proved the estimate $\|G(w,\tau)\|_{L_1(0,T;H^1_q(\mathbb{R}^n_+))} = \|G(w,\tau)\|_{\mathbb{G}^\#} \leq O(T)$ and $\|G(w,\tau)\|_{T,\mathbb{R}^n_+,\infty,q} \leq C$ in Lemma 2.12. The remaining term to estimate is $\|G(w,\tau)\|_{T,\mathbb{R}^n_+,r,q}$.

Let $0 < R_0, R_2, R_3, T_0$, $0 < R_1 < R_0$, $0 < T < T_0$, and $(w,\tau) \in \mathcal{K}(T, R_1, R_2)$. We denote by C a generic constant and by $O\colon \mathbb{R}_+ \to \mathbb{R}_+$ a generic function, with $O(t) \to 0$ for $t \to 0$, which may change from line to line, but is always independent of T, R_1, w, and τ.

By the proposition on embedding theorems (Proposition 1.14), we have

$$\|w\|_{L_\infty(0,T;W^1_\infty(\mathbb{R}^n_+))\cap L_\infty(0,T;H^1_q(\mathbb{R}^n_+))} + \|u_*\|_{L_\infty(0,T;W^1_\infty(\mathbb{R}^n_+))\cap L_\infty(0,T;H^1_q(\mathbb{R}^n_+))} + \|\tau\|_{L_\infty(0,T;L_\infty(\mathbb{R}^n_+))} \leq C_*.$$

By the mean value theorem, it follows that

$$\begin{aligned} \|G(w,\tau)\|_{T,\mathbb{R}^n_+,r,p} &= \|g(\nabla(w+u_*),\tau)\|_{T,\mathbb{R}^n_+,r,p} \\ &= \|g(\nabla(w+u_*),\tau) - g(0,0)\|_{T,\mathbb{R}^n_+,r,p} \\ &\leq \sup_{|\eta_1|,|\eta_2|<C_*} |(\nabla g)(\eta_1,\eta_2)|(\|\nabla(w+u_*)\|_{T,\mathbb{R}^n_+,r,p} + \|\tau\|_{T,\mathbb{R}^n_+,r,p}) \\ &\leq \sup_{|\eta_1|,|\eta_2|<C_*} |(\nabla g)(\eta_1,\eta_2)| T^{\frac{1}{r}}(\|\nabla(w+u_*)\|_{T,\mathbb{R}^n_+,\infty,p} + \|\tau\|_{T,\mathbb{R}^n_+,\infty,p}) \\ &\leq O(T). \end{aligned}$$

$\square$

We show, that we can choose T_0, R_0, R_2, R_3, $0 < R_1 < R_0$, and $0 < T < T_0$ such that

$$\Phi(\mathcal{K}(T, R_1, R_2, R_3)) \subset \mathcal{K}(T, R_1, R_2, R_3).$$

Fix $R_0 = T_0 = 1$. Let $(w, \tau) = \Phi(\tilde{w}, \tilde{\tau})$ with $(\tilde{w}, \tilde{\tau}) \in \mathcal{K}(T, R_1, R_2, R_3)$. By the maximal L_p-regularity of the Stokes problem (see Proposition 1.8), the proposition on the transport equation (Proposition 1.10), and the previous lemma, we conclude that

$$\begin{aligned}
&\|w\|_{{}_0\mathbb{E}_u(T,\mathbb{R}^n_+)} \\
&\quad = \|(\tilde{\Phi}_{0,\tau_0}(\tilde{w}, f_* + \operatorname{Div}(F_w(\tilde{w}) + F_\tau(\tilde{\tau})), G(\tilde{w}, \tilde{\tau}), H_\tau(\tilde{\tau})))_1\|_{{}_0\mathbb{E}_u(T,\mathbb{R}^n_+)} \\
&\quad \leq C\big(\|f_* + \operatorname{Div}(F_w(\tilde{w}) + F_\tau(\tilde{\tau}))\|_{\mathbb{F}_f(T,\mathbb{R}^n_+)} + \|H_\tau(\tilde{\tau})\|_{{}_0\mathbb{H}_u(T,\partial\mathbb{R}^n_+)}\big) \\
&\quad \leq C\big(\|f_*\|_{\mathbb{F}_f(T,\mathbb{R}^n_+)} + \|\operatorname{Div} F_w(\tilde{w})\|_{\mathbb{F}_f(T,\mathbb{R}^n_+)} + \|\operatorname{Div} F_\tau(\tilde{\tau})\|_{\mathbb{F}_f(T,\mathbb{R}^n_+)} + \|H_\tau(\tilde{\tau})\|_{{}_0\mathbb{H}_u(T,\partial\mathbb{R}^n_+)}\big) \\
&\quad \leq CR_1^2 + O(T),
\end{aligned}$$

and

$$\begin{aligned}
\|\tau\|_{L_\infty(0,T;H^1_q(\mathbb{R}^n_+))} &= \|(\tilde{\Phi}_{0,\tau_0}(\tilde{w}, f_* + F_w(\tilde{w}) + F_\tau(\tilde{\tau}), G(\tilde{w}, \tilde{\tau}), H_\tau(\tilde{\tau})))_2\|_{L_\infty(0,T;H^1_q(\mathbb{R}^n_+))} \\
&\leq C^{(1)}_{\mathrm{tra}}\big(\|\tau_0\|_{H^1_p(\mathbb{R}^n_+)} + \|G(\tilde{w}, \tilde{\tau})\|_{\mathbb{G}(T,\mathbb{R}^n_+)}\big) e^{C^{(1)}_{\mathrm{tra}} T^{1-\frac{1}{p}} \|\tilde{w}+u_*\|_{\mathbb{E}_u(T,\mathbb{R}^n_+)}} \\
&\leq C^{(1)}_{\mathrm{tra}}\big(\|\tau_0\|_{H^1_p(\mathbb{R}^n_+)} + O(T)\big) e^{O(T)},
\end{aligned}$$

as well as

$$\begin{aligned}
\|\partial_t \tau\|_{T,\mathbb{R}^n_+,r,p} &= \|(\tilde{\Phi}_{0,\tau_0}(\tilde{w}, f_* + F_w(\tilde{w}) + F_\tau(\tilde{\tau}), G(\tilde{w}, \tilde{\tau}), H_\tau(\tilde{\tau})))_2\|_{\widehat{W}^1_r(0,T;L_p(\mathbb{R}^n_+))} \\
&\leq \|G(\tilde{w}, \tilde{\tau})\|_{\mathbb{G}(T,\mathbb{R}^n_+)} + T^{\frac{1}{r}} \|\tilde{w} + u_*\|_{T,\Omega,\infty,\infty} \|\tau\|_{L_\infty(0,T;H^1_p(\mathbb{R}^n_+))} \\
&\leq O(T),
\end{aligned}$$

where $O\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a function with $O(t) \to 0$ for $t \to 0$, which is independent of R_1, $0 < R_1 < R_0$. Defining now $R_2 := 2C^{(1)}_{\mathrm{tra}}\|\tau_0\|_{H^1_p(\mathbb{R}^n_+)}$, $R_3 = 1$ and choosing $R_1, T > 0$ sufficiently small we find that $\Phi(\mathcal{K}(T, R_1, R_2, R_3))$ is contained in $\mathcal{K}(T, R_1, R_2, R_3)$.

The map Φ is a contraction

Next, we show that Φ is a contraction. In the proof of Theorem 2.6, we introduced for $n+2 < p < \infty$ the space $\mathbb{E}^w(T, \mathbb{R}^n_+) = \mathbb{E}^w_u(T, \mathbb{R}^n_+) \times \mathbb{E}^w_\tau(T, \mathbb{R}^n_+)$ with

$$\begin{aligned}
\mathbb{E}^w_u(T, \mathbb{R}^n_+) &= H^{\frac{1}{2}}_p(0, T; L_p(\mathbb{R}^n_+)) \cap L_p(0, T; H^1_p(\mathbb{R}^n_+)), \\
\mathbb{E}^w_\tau(T, \mathbb{R}^n_+) &= L_\infty(0, T; L_p(\mathbb{R}^n_+)).
\end{aligned}$$

We also introduced in the proof of this theorem the corresponding data spaces

$$\begin{aligned}
\mathbb{F}_f(T, \mathbb{R}^n_+) &= L_p(0, T; L_p(\mathbb{R}^n_+)), \\
\mathbb{G}^w(T, \mathbb{R}^n_+) &= L_\infty(0, T; L_p(\mathbb{R}^n_+)).
\end{aligned}$$

Let $(\tilde{w}_j, \tilde{\tau}_j) \in \mathcal{K}(T, R_1, R_2, R_3)$, $j = 1, 2$, and $(w_j, \tau_j) = \Phi(\tilde{w}_j, \tilde{\tau}_j)$ be the solutions of the linearized problem. We already proved that $(w_j, \tau_j) \in \mathcal{K}(T, R_1, R_2, R_3)$. Then, the difference

$$(w_{12}, \tau_{12}) := \Phi(\tilde{w}_2, \tilde{\tau}_2) - \Phi(\tilde{w}_1, \tilde{\tau}_1) = (w_2 - w_1, \tau_2 - \tau_1)$$

and the pressure difference $\pi_{12} = \pi_2 - \pi_1$ fulfill the equation

(2.43)

$$\begin{cases} \rho\partial_t w_{12} - \alpha\Delta w_{12} + \nabla\pi_{12} &= \operatorname{Div}(F_w(\tilde{w}_2) - F_w(\tilde{w}_1) + F_\tau(\tilde{\tau}_2) - F_\tau(\tilde{\tau}_1)) & \text{in } (0,T) \times \mathbb{R}^n_+, \\ \operatorname{div} u &= 0 & \text{in } (0,T) \times \mathbb{R}^n_+, \\ \partial_t \tau_{12} + (\tilde{w}_1 + u_*).\nabla\tau_{12} &= G(w_2, \tau_2) - G(w_1, \tau_1) + (\tilde{w}_2 - \tilde{w}_1).\nabla\tau_2 & \text{in } (0,T) \times \mathbb{R}^n_+, \\ -(u_{12,n}, \alpha(\partial_n u_{12,j})_{j=1}^n &= (0, H_\tau(\tilde{\tau}_2) - H_\tau(\tilde{\tau}_1)) & \text{on } (0,T) \times \partial\mathbb{R}^n_+, \\ u_{12}(0) &= 0 & \text{in } \mathbb{R}^n_+, \\ \tau_{12}(0) &= 0 & \text{in } \mathbb{R}^n_+. \end{cases}$$

In the following lemma, we summarize the estimates on the right-hand side of (2.43), which are relevant here. All this estimates are proved in Lemma 2.14.

Lemma 2.18. *Fix $n \in \mathbb{N}$, $n \geq 2$, $T_0, R_0, R_2, R_3 > 0$, and $n + 2 < p < \infty$. Assume that*

$$g \in C^1(\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}, \mathbb{R}^{n\times n}) \quad \textit{with} \quad g(0,0) = 0 \quad \textit{and} \quad \mu \in C^1(\mathbb{R}^{n\times n}, \mathbb{R}^{n\times n}).$$

Then, there exists a constant $C > 0$ and a function $O\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $O(t) \to 0$ for $t \to 0$ such that, for all $R_1 \in (0, R_0)$, $T \in (0, T_0)$, and $(w_j, \tau_j), (\tilde{w}_j, \tilde{\tau}_j) \in \mathcal{K}(T, R_1, R_2, R_3)$, $j = 1, 2$ the estimates

$$\begin{aligned} \|F^D_w(\tilde{w}_2) - F^D_w(\tilde{w}_1)\|_{\mathbb{F}_f(T,\mathbb{R}^n_+)} &\leq (CR_1 + O(T))\|\tilde{w}_2 - \tilde{w}_1\|_{\mathbb{E}^w_u(T,\mathbb{R}^n_+)}, \\ \|F^D_\tau(\tilde{\tau}_2) - F^D_\tau(\tilde{\tau}_1)\|_{\mathbb{F}_f(T,\mathbb{R}^n_+)} &\leq O(T)\|\tilde{\tau}_2 - \tilde{\tau}_1\|_{\mathbb{E}^w_\tau(T,\mathbb{R}^n_+)}, \\ \|G(\tilde{w}_2, \tilde{\tau}_2) - G(\tilde{w}_1, \tilde{\tau}_1)\|_{\mathbb{G}^w(T,\mathbb{R}^n_+)} &\leq O(T)(\|\tilde{w}_2 - \tilde{w}_1\|_{\mathbb{E}^w_u(T,\mathbb{R}^n_+)} + \|\tilde{\tau}_2 - \tilde{\tau}_1\|_{\mathbb{E}^w_\tau(T,\mathbb{R}^n_+)}), \\ \|(\tilde{w}_2 - \tilde{w}_1).\nabla\tau_2\|_{\mathbb{G}^w(T,\mathbb{R}^n_+)} &\leq O(T)\|\tilde{w}_2 - \tilde{w}_1\|_{\mathbb{E}^w_u(T,\mathbb{R}^n_+)} \end{aligned}$$

hold.

To apply Proposition 2.16, we need a relation between the right-hand side of the Stokes equation and the right-hand side on the boundary (see (2.35)). This relation follows due to the identity

$$(F^D_w(\tilde{w}_2) - F^D_w(\tilde{w}_1) + F^D_\tau(\tilde{\tau}_2) - F^D_\tau(\tilde{\tau}_1))_{j,n} = (H_\tau(\tilde{\tau}_2) - H_\tau(\tilde{\tau}_1))_j \quad \text{on } (0, T_0) \times \partial\mathbb{R}^n_+, \quad j = 1, \ldots, n-1.$$

By the proposition on the transport equation (Proposition 1.10), the proposition on the estimate of the velocity field in the weaker topology (Proposition 2.16), and the previous lemma, it follows that

$$\begin{aligned} &\|w_{12}\|_{\mathbb{E}^w_u(T,\mathbb{R}^n_+)} \\ &\quad = \|(\tilde{\Phi}_{0,0}(\tilde{w}_1, \operatorname{Div}(F^D_w(\tilde{w}_2) - F^D_w(\tilde{w}_1) + F^D_\tau(\tilde{\tau}_2) - F^D_\tau(\tilde{\tau}_1)), \\ &\qquad\quad G(\tilde{w}_2, \tilde{\tau}_2) - G(\tilde{w}_1, \tilde{\tau}_1) + (\tilde{w}_2 - \tilde{w}_1).\nabla\tau_2, H_\tau(\tilde{\tau}_2) - H_\tau(\tilde{\tau}_1)))_1\|_{\mathbb{E}^w_u(T,\mathbb{R}^n_+)} \\ &\quad \leq C\|F^D_w(\tilde{w}_2) - F^D_w(\tilde{w}_1) + F^D_\tau(\tilde{\tau}_1) - F^D_\tau(\tilde{\tau}_1)\|_{\mathbb{F}_f(T,\mathbb{R}^n_+)} \\ &\quad \leq C\big(\|F^D_w(\tilde{w}_2) - F^D_w(\tilde{w}_1)\|_{\mathbb{F}_f(T,\mathbb{R}^n_+)} + \|F^D_\tau(\tilde{\tau}_2) - F^D_\tau(\tilde{\tau}_1)\|_{\mathbb{F}_f(T,\mathbb{R}^n_+)}\big) \\ &\quad \leq (CR_1 + O(T))\big(\|\tilde{w}_2 - \tilde{w}_1\|_{\mathbb{E}^w_u(T,\mathbb{R}^n_+)} + \|\tilde{\tau}_2 - \tilde{\tau}_1\|_{\mathbb{E}^w_\tau(T,\mathbb{R}^n_+)}\big), \end{aligned}$$

and

$$
\begin{aligned}
&\|\tau_{12}\|_{\mathbb{E}^w_\tau(T,\mathbb{R}^n_+)} \\
&\quad = \|(\tilde{\Phi}_{0,0}(\tilde{w}_1, \operatorname{Div}(F^D_w(\tilde{w}_2) - F^D_w(\tilde{w}_1) + F^D_\tau(\tilde{\tau}_2) - F^D_\tau(\tilde{\tau}_1)), \\
&\qquad\quad G(\tilde{w}_2, \tilde{\tau}_2) - G(\tilde{w}_1, \tilde{\tau}_1) + (\tilde{w}_2 - \tilde{w}_1).\nabla\tau_2, H_\tau(\tilde{\tau}_2) - H_\tau(\tilde{\tau}_1)))_2\|_{\mathbb{E}^w_\tau(T,\mathbb{R}^n_+)} \\
&\quad \le \|G(\tilde{w}_2, \tilde{\tau}_2) - G(\tilde{w}_1, \tilde{\tau}_1) + (\tilde{w}_2 - \tilde{w}_1).\nabla\tau_2\|_{\mathbb{G}^w(T,\mathbb{R}^n_+)} \\
&\quad \le (\|G(\tilde{w}_2, \tilde{\tau}_2) - G(\tilde{w}_1, \tilde{\tau}_1)\|_{\mathbb{G}^w(T,\mathbb{R}^n_+)} + \|(\tilde{w}_2 - \tilde{w}_1).\nabla\tau_2\|_{\mathbb{G}^w(T,\mathbb{R}^n_+)}) \\
&\quad \le O(T)\big(\|\tilde{w}_2 - \tilde{w}_1\|_{\mathbb{E}^w_u(T,\mathbb{R}^n_+)} + \|\tilde{\tau}_2 - \tilde{\tau}_1\|_{\mathbb{E}^w_\tau(T,\mathbb{R}^n_+)}\big).
\end{aligned}
$$

Choosing $0 < R_1, T$ sufficiently small, it follows the contraction property in $\mathbb{E}^w(T, \mathbb{R}^n_+)$, i.e.

$$
\begin{aligned}
\|\Phi(\tilde{w}_2, \tilde{w}_2) - \Phi(\tilde{w}_1, \tilde{w}_1)\|_{\mathbb{E}^w(T,\mathbb{R}^n_+)} &= \|w_{12}\|_{\mathbb{E}^w_u(T,\mathbb{R}^n_+)} + \|\tau_{12}\|_{\mathbb{E}^w_\tau(T,\mathbb{R}^n_+)} \\
&\le \frac{1}{2}\big(\|\tilde{w}_2 - \tilde{w}_1\|_{\mathbb{E}^w_u(T,\mathbb{R}^n_+)} + \|\tilde{\tau}_2 - \tilde{\tau}_1\|_{\mathbb{E}^w_\tau(T,\mathbb{R}^n_+)}\big).
\end{aligned}
$$

Application of the fixed point argument and completion of the proof

Application of Proposition 1.13 proves the existence of a unique fixed point of Φ or equivalent the existence of a unique solution

$$
(u, \tau) \in H^1_p(0, T; L_p(\mathbb{R}^n_+)) \cap L_p(0, T; H^2_q(\mathbb{R}^n_+)) \times \widehat{W}^1_r(0, T; L_p(\mathbb{R}^n_+)) \cap L_\infty(0, T; H^1_p(\mathbb{R}^n_+))
$$

of (2.19). The corresponding pressure is defined by

$$
\nabla\pi = \big(1 - P_p\big)\big(\operatorname{Div}\mu(\tau) - f - \rho(\partial_t u + u.\nabla u) + \alpha\Delta u\big) \in L_p(0, T; L_p(\mathbb{R}^n_+)).
$$

It remains to show that τ admits more time regularity. The function τ in particular fulfills the transport equation

$$
\partial_t\tau + (w + u_*).\nabla\tau = G(w, \tau).
$$

In Lemma 2.17 we proved that $G(w, \tau) \in L_\infty(0, T; L_p(\mathbb{R}^n_+))$ and hence, $\partial_t\tau \in L_\infty(0, T; L_p(\mathbb{R}^n_+))$, by the proposition on the transport equation (Proposition 1.10). □

Chapter 3

Generalized Newtonian two-phase flow

The aim of this chapter is the investigation of a free boundary problem describing the motion of two incompressible generalized Newtonian fluids. The domain occupied by the first fluid is denoted by $\Omega_-(t) \subset \mathbb{R}^n$ and the domain occupied by the second fluid is denoted by $\Omega_+(t) \subset \mathbb{R}^n$. We assume that the fluids are separated by a free interface, which we denote by $\Gamma(t)$. At the initial configuration, we assume that the two fluids are separated by an interface Γ_0, which is given as a graph over a height function h_0, i.e.

$$\Gamma_0 = \{(x', x_n) \in \mathbb{R}^n \colon x_n = h_0(x')\} \quad \text{and} \quad \Omega_\pm(0) = \{(x', x_n) \colon \pm (x_n - h_0(x')) > 0\}.$$

The domain occupied by the fluids is denoted by $\Omega(t) = \Omega_+(t) \cup \Omega_-(t)$ and the normal, pointing from $\Omega_-(t)$ to $\Omega_+(t)$ by $\nu(t)$. The system under consideration reads:

$$\left\{\begin{array}{rcll} \rho(\partial_t u + u.\nabla u) - \operatorname{Div} 2\alpha(|Eu|^2)Eu + \nabla\pi &=& -\rho\gamma_e e_n & \text{in } (0,T_0)\times\Omega(t), \\ \operatorname{div} u &=& 0 & \text{in } (0,T_0)\times\Omega(t), \\ -[\![2\alpha(|Eu|^2)Eu - \pi]\!]\nu &=& \sigma\kappa\nu & \text{on } (0,T_0)\times\Gamma(t), \\ [\![u]\!] &=& 0 & \text{on } (0,T_0)\times\Gamma(t), \\ V &=& u\cdot\nu & \text{on } (0,T_0)\times\Gamma(t), \\ u(0) &=& u_0 & \text{in } \mathbb{R}^n \setminus \Gamma_0, \\ \Gamma(0) &=& \Gamma_0. & \end{array}\right. \tag{3.1}$$

This system will be described as follows: The unknowns of this system are the velocity field u, the pressure π, and the interface Γ. The velocity u and the pressure π consist of the velocities and pressures of both fluids, i.e.

$$(u,\pi) = (u_+,\pi_+)\chi_{\Omega_+} + (u_-,\pi_-)\chi_{\Omega_-},$$

where $u_\pm$ and $\pi_\pm$ are the velocity field and the pressure of the fluid occupying $\Omega_\pm$ respectively. The given and constant density of the fluid occupying $\Omega_\pm$ is denoted by $\rho_\pm > 0$ and the given viscosity function by $\alpha_\pm \colon [0,\infty) \to [0,\infty)$. We apply the same convention for the density and the viscosity function, i.e.

$$(\rho,\alpha) = (\rho_+,\alpha_+)\chi_{\Omega_+} + (\rho_-,\alpha_-)\chi_{\Omega_-}.$$

The symmetric part of the velocity gradient is denoted by $Eu = \frac{1}{2}(\nabla u + (\nabla u)^T)$ and the jump of a quantity f at the interface is denoted by $[\![f]\!] = \gamma_+ f - \gamma_- f$, where $\gamma_\pm$ denotes the upper and lower trace on $\Gamma(t)$. In the case that two traces coincide, we write $\gamma = \gamma_+$. The normal velocity of the interface is denoted by V, the mean curvature by κ, the given and constant acceleration of gravity by $\gamma_a > 0$, and the given and constant surface tension by σ.

Further, the initial velocity field u_0 and the initial interface $\Gamma_0 = \operatorname{graph} h_0$, which is given as a graph of a height function h_0, are given. We assume that the initial velocity satisfies the natural compatibility conditions

$$\operatorname{div} u_0 = 0 \quad \text{in } \mathbb{R}^n \setminus \Gamma_0 \quad \text{and} \quad ([\![\alpha(|Eu_0|^2)Eu_0]\!]\nu_0)_{\tan} = 0, \quad [\![u_0]\!] = 0 \quad \text{on } \Gamma_0. \tag{3.2}$$

An initial value of the pressure jump $[\![\pi]\!]$ is implicitly given by

$$[\![\pi_0]\!] := 2([\![\alpha(|Eu_0|^2)Eu_0]\!]\nu_0) \cdot \nu_0 + \sigma\kappa \quad \text{on } \Gamma_0. \tag{3.3}$$

The first equation of (3.1) describes the balance of momentum, assuming the stress tensor has the form

$$\mathcal{S}(u, \pi) = 2\alpha(|Eu|^2)Eu - \pi,$$

and the only external force is the gravity. The divergence free condition describes incompressibility of both fluids, since the densities are constant in both domains $\Omega_\pm(t)$. The first boundary condition at the interface says, that the tangential part of the normal component of the stress $([\![\mathcal{S}(u,\pi)]\!]\nu)_{\tan}$ is continuous, and the jump of the normal part $([\![\mathcal{S}(u,\pi)]\!]\nu)\cdot\nu$ is proportional to the mean curvature κ. The zero jump condition of the velocity field on the free interface gives the continuity of the velocity field on the interface. The kinematic boundary condition ($V = u \cdot \nu$ on $(0, T_0) \times \Gamma(t)$) couples the interface to the fluid and says that the interface is transported by the motion of the fluid.

Systems similar to (3.1) have been studied intensively in the literature. There are various results regarding the one-phase flow problem in the Newtonian case ($\alpha > 0$ is constant)

$$\left\{\begin{array}{rcll} \rho(\partial_t u + u.\nabla u) - 2\alpha \operatorname{Div} Eu + \nabla\pi & = & -\rho\gamma_e e_n & \text{in } (0,T_0) \times \Omega(t), \\ \operatorname{div} u & = & 0 & \text{in } (0,T_0) \times \Omega(t), \\ -(2\alpha Eu - \pi)\nu & = & \sigma\kappa\nu & \text{on } (0,T_0) \times \Gamma(t), \\ V & = & u \cdot \nu & \text{on } (0,T_0) \times \Gamma(t), \\ u(0) & = & u_0 & \text{in } \mathbb{R}^n \setminus \Gamma_0, \\ \Gamma(0) & = & \Gamma_0, & \end{array}\right.$$

using the formulation of this system in Lagrangian coordinates. For an overview over the existing literature, we refer to Chapter 4. Also the two-phase problem, where both fluids are of Newtonian type was investigated in Lagrangian framework by Denisova [Den90, Den94], Denisova and Solonnikov [DS95] as well as Tanaka [Tan95].

The investigation of the one-phase problem in Eulerian coordinates, using the Hanzawa transformation to reduce the problem on a fixed domain goes back to the work of Beale [Bea84] and Beale and Nishida [BN85]. They discussed an ocean like domain, i.e. a domain bounded by a solid from below and a free surface from above, using L_2-theory. Recently, also applying the Hanzawa transformation, Denk, Geissert, Hieber, Saal and Sawada [DGH$^+$11] considered the one-phase problem in Eulerian coordinates in the L_p-setting. They analysed the spin coating process for a Newtonian fluid in an ocean like domain. Rotational and wetting effects on the solid below were also incorporated. The two-phase case, where both fluids are Newtonian was analysed by Prüß and Simonett [PS09, PS10, PS11, PS] in the L_p-setting. They considered a model, where both fluids initially occupy a domain close to half spaces and proved local-in-time solvability for arbitrarily large initial data as well as large time well-posedness for of small initial data, with and without the effect of gravity. Moreover, they gave a rigorous proof for the Rayleigh-Taylor instability, i.e. they proved that the system is L_p-unstable, provided that a fluid with a larger density is positioned on top of a fluid with the lower density.

In the one-phase case, were a Newtonian fluid occupies an ocean like domain, Bae [Bae11] was able to show the existence of global strong solutions, applying energy methods.

Abels [Abe07a, Abe07b] and Plotnikov [Plo93] investigated the two-phase flow problem for a generalized Newtonian fluid on the existence of varifold-solutions, using energy methods and monotone operator theory.

The aim of this chapter is to prove the existence of a unique strong solutions of (3.1) in the L_p-setting.

Before we formulate our main result, we recall some definitions. For a vector $x \in \mathbb{R}^n$, we write $x = (x', x_n) \in \mathbb{R}^n$ with $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. Moreover, we define

$$\mathbb{R}^n_\pm = \{(x', x_n) \in \mathbb{R}^n \colon \pm x_n > 0\} \quad \text{and} \quad \dot{\mathbb{R}}^n = \{(x', x_n) \in \mathbb{R}^n \colon x_n \neq 0\}.$$

The gradient and the Laplace operator with respect to $x' \in \mathbb{R}^{n-1}$ is denoted by ∇' and Δ' respectively. Furthermore, we define the Hanzawa transformation. We fix $T_0 > 0$ and we assume that for $0 < t < T_0$ the interface $\Gamma(t) = \operatorname{graph}(h(t))$ is given as the graph over a hight function $h(t) \colon \mathbb{R}^{n-1} \to \mathbb{R}$. Then, the Hanzawa transformation is defined by

$$\Theta_h \colon (0, T_0) \times \dot{\mathbb{R}}^n \to \bigcup_{t \in (0,T_0)} \{t\} \times \Omega(t), \quad \Theta_h(t, x', x_n) = (t, x', x_n + h(t, x')).$$

Large time solvability of (3.1) is established by our next theorem.

Theorem 3.1. *Fix $n + 2 < p < \infty$, $T_0 > 0$, and $\rho_\pm, \gamma_e, \sigma > 0$. Let $\alpha_\pm \in C^3([0, \infty))$ with $\alpha_\pm(0) > 0$. Then, there exists $\varepsilon > 0$, such that for all $h_0 \in W_p^{3-\frac{2}{p}}(\mathbb{R}^{n-1})$, $\Gamma_0 = \operatorname{graph}(h_0)$, and $u_0 \in W_p^{2-\frac{2}{p}}(\mathbb{R}^n \setminus \Gamma_0)$, satisfying the compatibility conditions*

$$\operatorname{div} u_0 = 0 \quad \text{in } \mathbb{R}^n \setminus \Gamma_0 \quad \text{and} \quad \left([\![\alpha(|Eu_0|^2) Eu_0]\!]\nu_0\right)_{tan} = 0, \quad [\![u_0]\!] = 0 \quad \text{on } \Gamma_0,$$

where $\nu_0 = \frac{1}{\sqrt{1+|\nabla' h|^2}}(-\nabla' h, 1)^T$ denotes the normal on Γ_0, as well as the smallness condition

$$\|u_0\|_{W_p^{2-\frac{2}{p}}(\mathbb{R}^n \setminus \Gamma_0)} + \|h_0\|_{W_p^{3-\frac{2}{p}}(\mathbb{R}^{n-1})} < \varepsilon,$$

there exists a solution (u, π, Γ) on $(0, T_0)$ of (3.1). Furthermore, $\Gamma(t)$ is for $t \in (0, T_0)$ given as a graph $\Gamma(t) = \operatorname{graph}(h(t))$ with

$$h \;\in\; W_p^{2-\frac{1}{2p}}(0, T_0; L_p(\mathbb{R}^{n-1})) \cap H_p^1(0, T_0; W_p^{2-\frac{1}{p}}(\mathbb{R}^{n-1})) \cap L_p(0, T_0; W_p^{3-\frac{1}{p}}(\mathbb{R}^{n-1})).$$

This solution is in the regularity class

$$\begin{aligned}
u \circ \Theta_h^{-1} \;&\in\; H_p^1(0, T_0; L_p(\mathbb{R}^n)) \cap L_p(0, T_0; H_p^2(\dot{\mathbb{R}}^n)), \\
\pi \circ \Theta_h^{-1} \;&\in\; L_p(0, T_0; \widehat{H}_p^1(\dot{\mathbb{R}}^n))), \\
[\![\pi \circ \Theta_h^{-1}]\!] \;&\in\; W_p^{\frac{1}{2}-\frac{1}{2p}}(0, T_0; L_p(\mathbb{R}^{n-1})) \cap L_p(0, T_0; W_p^{1-\frac{1}{p}}(\mathbb{R}^{n-1})).
\end{aligned}$$

Moreover, the solution is unique in this regularity class.

3.1 Proof of the main theorem

A proof of Theorem 3.1 is given here. First, let us sketch the main ideas of the proof.

Sketch of the proof

We proceed the following way: First, we absorb the gravity term in the pressure and apply the Hanzawa transformation to reduce (3.1) to a problem on a fixed domain. Next, we linearize the transformed problem around the trivial equilibrium and rewrite it in the form of a fixed point problem. Then, we investigate the nonlinearities arising from the Hanzawa transformation and linearization. By the solvability result of the associated linearization of Prüß and Simonett [PS11, Theorem 3.1] and the mapping properties of the nonlinearities, we are in a position to apply the contraction mapping principle to solve the nonlinear problem in a last step.

Proof of Theorem 3.1. In a first step, we reduce (3.1) on a fixed domain.

Hanzawa transformation

In order to investigate problem (3.1), we will apply transformations. First of all, it should be noted that the gravity term $-\rho\gamma_a e_n$ admits a potential $x_n \mapsto \rho\gamma_a x_n$, since the density in $\Omega_\pm(t)$ is constant, and thus may be absorbed in the pressure term. Hence, we introduce a new pressure function

$$\pi_1(t,x',x_n) = \pi(t,x',x_n) + \rho_\pm\gamma_a x_n, \quad (t,x',x_n) \in (0,T_0)\times\Omega_\pm(t), \tag{3.4}$$

which results in an additional boundary term. The new system reads:

$$\left\{\begin{array}{rcll} \rho(\partial_t u + u.\nabla u) - \operatorname{Div} 2\alpha(|Eu|^2)Eu + \nabla\pi_1 &=& 0 & \text{in } (0,T_0)\times\Omega(t),\\ \operatorname{div} u &=& 0 & \text{in } (0,T_0)\times\Omega(t),\\ -[\![2\alpha(|Eu|^2)Eu - \pi_1 + \rho\gamma_a x_n]\!]\nu &=& \sigma\kappa\nu & \text{on } (0,T_0)\times\Gamma(t),\\ [\![u]\!] &=& 0 & \text{on } (0,T_0)\times\Gamma(t),\\ V &=& u\cdot\nu & \text{on } (0,T_0)\times\Gamma(t),\\ u(0) &=& u_0 & \text{in } \mathbb{R}^n\setminus\Gamma_0,\\ \Gamma(0) &=& \Gamma_0. & \end{array}\right. \tag{3.5}$$

To receive a problem on the fixed domain $\dot{\mathbb{R}}^n = \{(x',x_n)\in\mathbb{R}^n : x_n \neq 0\}$ instead of $\Omega(t)$, we apply the Hanzawa transformation:

$$\Theta_h\colon (0,T_0)\times\dot{\mathbb{R}}^n \to \bigcup_{t\in(0,T_0)} \{t\}\times\Omega(t), \quad (t,x',x_n)\mapsto (t,x',h(t,x')+x_n).$$

From now on, let $(t,x',x_n)\in(0,T_0)\times\dot{\mathbb{R}}^n$. We denote the transformed velocity function by

$$(v'(t,x',x_n), v_n(t,x',x_n)) = v(t,x',x_n) := (u\circ\Theta_h)(t,x',x_n) = u(t,x',h(t,x')+x_n),$$

where we split the transformed velocity field $v = (v', v_n)$ in a tangential part $v'(t,x',x_n)\in\mathbb{R}^{n-1}$ and a normal part $v_n(t,x',x_n)\in\mathbb{R}$. The transformed pressure is denoted by

$$\theta(t,x',x_n) := (\pi\circ\Theta_h)(t,x',x_n) = \pi_1(t,x',h(t,x')+x_n).$$

By the chain rule, we calculate for $k, l, m = 1, \dots, n$

$$
\begin{aligned}
(\partial_t u_k) \circ \Theta_h &= \partial_t v_k - \partial_n v_k \partial_t h, \\
(\partial_m u_k) \circ \Theta_h &= \partial_m v_k - \partial_n v_k \partial_m h, \\
(\partial_l \partial_m u_k) \circ \Theta_h &= \partial_l \partial_m v_k - \partial_m \partial_n v_k \partial_l h - \partial_l \partial_n v_k \partial_m h + \partial_n^2 v_k \partial_l h \partial_m h - \partial_n v_k \partial_l \partial_m h.
\end{aligned} \tag{3.6}
$$

It is worth pointing out that $\partial_n h = 0$. In particular, the Laplace operator transforms to

$$
(\Delta u_k) \circ \Theta_h = \Delta v_k - 2(\partial_n \nabla' v_k) \cdot \nabla' h + \partial_n^2 v_k |\nabla' h|^2 - \partial_n v_k \Delta' h, \quad k = 1, \dots, n. \tag{3.7}
$$

The transformed symmetric gradient has the form

$$
\begin{aligned}
\mathcal{E}(v, h) := (Eu) \circ \Theta_h &= Ev - \frac{1}{2}(\partial_n v (\nabla h)^T + \nabla h (\partial_n v)^T) \\
&= Ev - \frac{1}{2} \begin{pmatrix} \partial_n v' (\nabla' h)^T + \nabla' h (\partial_n v')^T & \partial_n v_n \nabla' h \\ \partial_n v_n (\nabla' h)^T & 0 \end{pmatrix}.
\end{aligned} \tag{3.8}
$$

We are now in a position to transform the terms in first equations of (3.5), the balance of momentum. Applying the Hanzawa transformation gives

$$
\rho(\partial_t u + u.\nabla u) \circ \Theta_h - (\operatorname{Div} 2\alpha(|Eu|^2) Eu) \circ \Theta_h + (\nabla \pi_1) \circ \Theta_h = 0 \quad \text{in } (0, T_0) \times \dot{\mathbb{R}}^n. \tag{3.9}
$$

This equation is the next subject. First, we recall the definition of the quasilinear operator $\mathcal{A}$ defined in the Subsection on the generalized Stokes operator (Subsection 1.2.3), i.e.

$$
[\mathcal{A}(E\tilde{u})\tilde{v}]_j = - \sum_{k,l,m=1}^{n} \mathcal{A}_{j,k}^{l,m}(E\tilde{u}) \partial_l \partial_m \tilde{v}_k, \quad j = 1, \dots, n
$$

with

$$
\mathcal{A}_{j,k}^{l,m}(E\tilde{u}) = \alpha(|E\tilde{u}|^2)(\delta_{l,m}\delta_{j,k} + \delta_{j,m}\delta_{k,l}) + 4\alpha'(|E\tilde{u}|^2)(E\tilde{u})_{j,l}(E\tilde{u})_{k,m}, \quad j, k, l, m = 1, \dots, n,
$$

as well as the properties $\mathcal{A}(Eu)u = -\operatorname{Div} \alpha(|Eu|^2) Eu$ and $\mathcal{A}(0)u = -\alpha(0)\Delta u$ (see (1.4)). Hence, the transformed equation (3.9) can be written in the form

$$
\rho \partial_t v - \alpha(0)\Delta v + \nabla \theta = F(v, \theta, h) \quad \text{in } (0, T_0) \times \dot{\mathbb{R}}^n,
$$

with

$$
\begin{aligned}
&F(v, \theta, h) \\
&\quad := \big(\mathcal{A}(0)v - (\mathcal{A}(Eu)u) \circ \Theta_h\big) + \big(\rho \partial_t v - \rho(\partial_t u) \circ \Theta_h\big) - \rho(u.\nabla u) \circ \Theta_h + \big(\nabla \theta - (\nabla \pi_1) \circ \Theta_h\big).
\end{aligned}
$$

We simplify each of these terms. By (3.6) and (3.7), it follows that

$$
\begin{aligned}
&(\mathcal{A}(0)v)_j - (\mathcal{A}(Eu)u)_j \circ \Theta_h \\
&\quad = \sum_{k,l,m=1}^{n} (\mathcal{A}_{j,k}^{l,m}(Eu)\partial_l\partial_m u_k) \circ \Theta_h - \sum_{j,k,l=1}^{n} \mathcal{A}_{j,k}^{l,m}(0)\partial_l\partial_m v_k \\
&\quad = \sum_{k,l,m=1}^{n} \big(\mathcal{A}_{j,k}^{l,m}(\mathcal{E}(v,h)) - \mathcal{A}_{j,k}^{l,m}(0)\big)\big((\partial_l\partial_m u_k) \circ \Theta_h\big) + \sum_{j,k,l=1}^{n} \mathcal{A}_{j,k}^{l,m}(0)\big((\partial_l\partial_m u_k) \circ \Theta_h - \partial_l\partial_m v_k\big) \\
&\quad = \Big(\sum_{k,l,m=1}^{n} \big(\mathcal{A}_{j,k}^{l,m}(\mathcal{E}(v,h)) - \mathcal{A}_{j,k}^{l,m}(0)\big)\big((\partial_l\partial_m u_k) \circ \Theta_h\big)\Big) + \alpha(0)\big((\Delta u_j) \circ \Theta_h - \Delta v_j\big) \\
&\quad = \sum_{k,l,m} \big(\mathcal{A}_{j,k}^{l,m}(\mathcal{E}(v,h)) - \mathcal{A}_{j,k}^{l,m}(0)\big) \times \\
&\qquad\quad \times \big(\partial_l\partial_m v_k - \partial_m\partial_n v_k \partial_l h - \partial_l\partial_n v_k \partial_m h + \partial_n^2 v_k \partial_l h \partial_m h - \partial_n v_k \partial_l\partial_m h\big) \\
&\qquad - \alpha(0)\big(2(\partial_n \nabla' v_j) \cdot \nabla' h - \partial_n^2 v_j |\nabla' h|^2 + \partial_n v_j \Delta' h\big), \quad j = 1, \ldots, n.
\end{aligned}
$$

Further, taking into account (3.6), we compute

$$
\big(\rho\partial_t v - \rho(\partial_t u) \circ \Theta_h\big) = \rho\partial_n v \partial_t h,
$$

and

$$
\big(\nabla\theta - (\nabla\pi_1) \circ \Theta_h\big) = \partial_n\theta\nabla h,
$$

as well as

$$
-\rho(u.\nabla u) \circ \Theta_h = -\rho\big(v.\nabla v - \sum_{k=1}^{n} v_k \partial_n v \partial_k h\big) = -\rho\big(v.\nabla v - (v' \cdot \nabla' h)\partial_n v\big).
$$

In summary, we have

$$
\begin{aligned}
F(v,\theta,h)_j = & \sum_{k,l,m} \big(\mathcal{A}_{j,k}^{l,m}(\mathcal{E}(v,h)) - \mathcal{A}_{j,k}^{l,m}(0)\big) \times \\
& \quad \times \big(\partial_l\partial_m v_k - \partial_m\partial_n v_k \partial_l h - \partial_l\partial_n v_k \partial_m h + \partial_n^2 v_k \partial_l h \partial_m h - \partial_n v_k \partial_l\partial_m h\big) \\
& - \alpha(0)\big(2(\partial_n \nabla' v_j) \cdot \nabla' h - \partial_n^2 v_j |\nabla' h|^2 + \partial_n v_j \Delta' h\big) + \rho\partial_n v_j \partial_t h \\
& - \rho\big(v.\nabla v_j - (v' \cdot \nabla' h)\partial_n v_j\big) + \partial_n\theta\partial_j h, \quad j = 1, \ldots, n.
\end{aligned} \tag{3.10}
$$

Applying the Hanzawa transformation to the divergence free condition gives

$$
(\operatorname{div} u) \circ \Theta_h = 0 \quad \text{in } (0,T_0) \times \dot{\mathbb{R}}^n.
$$

This can be written in the form

$$
\operatorname{div} v = F_d(v,h) \quad \text{in } (0,T_0) \times \dot{\mathbb{R}}^n,
$$

with the nonlinear right-hand side

$$
F_d(v,h) = \operatorname{div} v - (\operatorname{div} u) \circ \Theta_h = (\partial_n v) \cdot \nabla h, \tag{3.11}
$$

by (3.6).

To transform the boundary condition

$$-[\![2\alpha(|Eu|^2)Eu - \pi_1 + \rho\gamma_a x_n]\!]\nu = \sigma\kappa\nu \quad \text{on } (0,T_0)\times\Gamma(t),$$

it is convenient to calculate first $\mathcal{E}(v,h)\nu$. Since the hypersurface is given by local coordinates, the normal field can be represented by

$$\nu = \frac{1}{\sqrt{1+|\nabla' h|^2}}(-\nabla' h, 1)^T.$$

We compute

$$\begin{aligned}
Ev\nu &= \frac{1}{2\sqrt{1+|\nabla' h|^2}}\begin{pmatrix} \nabla' v' + (\nabla' v')^T & \nabla' v_n + \partial_n v' \\ (\nabla' v_n)^T + (\partial_n v')^T & 2\partial_n v_n \end{pmatrix}\begin{pmatrix} -\nabla' h \\ 1 \end{pmatrix} \\
&= \frac{1}{2\sqrt{1+|\nabla' h|^2}}\left(\begin{pmatrix} \partial_n v' + \nabla' v_n \\ 2\partial_n v_n \end{pmatrix} - \begin{pmatrix} (\nabla' v' + (\nabla' v')^T)\nabla' h \\ \nabla' v_n \cdot \nabla' h + \partial_n v' \cdot \nabla' h \end{pmatrix}\right),
\end{aligned}$$

and, taking into account (3.8), we infer

$$\begin{aligned}
&\mathcal{E}(v,h)\nu \\
&= \frac{1}{2\sqrt{1+|\nabla' h|^2}}\left(2Ev\begin{pmatrix} -\nabla' h \\ 1 \end{pmatrix} - \begin{pmatrix} \partial_n v'(\nabla' h)^T + \nabla' h(\partial_n v')^T & \partial_n v_n \nabla' h \\ \partial_n v_n (\nabla' h)^T & 0 \end{pmatrix}\begin{pmatrix} -\nabla' h \\ 1 \end{pmatrix}\right) \\
&= \frac{1}{2\sqrt{1+|\nabla' h|^2}}\left(\begin{pmatrix} \partial_n v' + \nabla' v_n \\ 2\partial_n v_n \end{pmatrix}\right. \\
&\quad \left. - \begin{pmatrix} (\nabla' v' + (\nabla' v')^T)\nabla' h - |\nabla' h|^2\partial_n v' - (\partial_n v' \cdot \nabla' h)\nabla' h + \partial_n v_n \nabla' h \\ \nabla' v_n \cdot \nabla' h + \partial_n v' \cdot \nabla' h - \partial_n v_n |\nabla' h|^2. \end{pmatrix}\right).
\end{aligned} \tag{3.12}$$

Moreover, the curvature is given by (see for instance [PS10, equation (2.5)])

$$\kappa = \frac{\Delta' h}{\sqrt{1+|\nabla' h|^2}} - \frac{\nabla' h \cdot (\nabla'^2 h \nabla' h)}{(1+|\nabla' h|^2)^{\frac{3}{2}}} = \Delta' h - G_\kappa(h),$$

with

$$G_\kappa(h) = \big(1 - \frac{1}{\sqrt{1+|\nabla' h|^2}}\big)\Delta' h + \frac{\nabla' h \cdot (\nabla'^2 h \nabla' h)}{(1+|\nabla' h|^2)^{\frac{3}{2}}}. \tag{3.13}$$

Thus, the boundary condition

$$-[\![2\alpha(|Eu|^2)Eu - \pi_1 + \rho\gamma_a x_n]\!]\nu = \sigma\kappa\nu \quad \text{on } (0,T_0)\times\Gamma(t)$$

transforms to

$$-[\![2\alpha(|\mathcal{E}(v,h)|^2)\mathcal{E}(v,h) - \theta + \rho\gamma_a h]\!]\nu = \sigma(\Delta' h - G_\kappa(h))\nu \quad \text{on } (0,T_0)\times\mathbb{R}^{n-1}. \tag{3.14}$$

Multiplying with $\sqrt{1+|\nabla' h|^2}$ and decomposing into horizontal and vertical components, we deduce that

$$\begin{aligned}
0 &= \sqrt{1+|\nabla' h|^2}\big(-[\![\theta]\!]\nu' + [\![\rho\gamma_a h]\!]\nu' + \sigma\kappa\nu'\big) + \sqrt{1+|\nabla' h|^2}\big([\![2\alpha(|\mathcal{E}(v,h)|^2)\mathcal{E}(v,h)]\!]\nu\big)' \\
&= \sqrt{1+|\nabla' h|^2}\big(-[\![\theta]\!]\nu' + [\![\rho\gamma_a h]\!]\nu' + \sigma\kappa\nu'\big) + \sqrt{1+|\nabla' h|^2}\big([\![2\alpha(|\mathcal{E}(v,h)|^2)\mathcal{E}(v,h)]\!]\nu\big)' \\
&\quad + [\![\alpha(0)\partial_n v']\!] + [\![\alpha(0)\nabla' v_n]\!] - [\![\alpha(0)\partial_n v']\!] - [\![\alpha(0)\nabla' v_n]\!] \\
&= [\![\alpha(0)\partial_n v']\!] + [\![\alpha(0)\nabla' v_n]\!] + \tilde{H}_v'(v,[\![\theta]\!],h),
\end{aligned}$$

with

$$\begin{aligned}
\tilde{H}_v'(v,[\![\theta]\!],h) = -[\![\alpha(0)\partial_n v']\!] - [\![\alpha(0)\nabla' v_n]\!] &+ \sqrt{1+|\nabla' h|^2}\big([\![2\alpha(|\mathcal{E}(v,h)|^2)\mathcal{E}(v,h)]\!]\nu\big)' \\
&+ \sqrt{1+|\nabla' h|^2}\big(-[\![\theta]\!]\nu' + [\![\rho\gamma_a h]\!]\nu' + \sigma\kappa\nu'\big),
\end{aligned}$$

as well as

$$\begin{aligned}
0 &= \sqrt{1+|\nabla' h|^2}\big(-[\![\theta]\!]\nu_n + [\![\rho\gamma_a h]\!]\nu_n + \sigma\kappa\nu_n\big) + \sqrt{1+|\nabla' h|^2}\big([\![2\alpha(|\mathcal{E}(v,h)|^2)\mathcal{E}(v,h)]\!]\nu\big)_n \\
&= -[\![\theta]\!] + [\![\rho\gamma_a h]\!] + \sigma(\Delta' h - G_\kappa(h)) + \sqrt{1+|\nabla' h|^2}\big([\![2\alpha(|\mathcal{E}(v,h)|^2)\mathcal{E}(v,h)]\!]\nu\big)_n \\
&= -[\![\theta]\!] + [\![\rho\gamma_a h]\!] + \sigma(\Delta' h - G_\kappa(h)) + \sqrt{1+|\nabla' h|^2}\big([\![2\alpha(|\mathcal{E}(v,h)|^2)\mathcal{E}(v,h)\nu]\!]\nu\big)_n \\
&\quad + 2[\![\alpha(0)\partial_n v_n]\!] - 2[\![\alpha(0)\partial_n v_n]\!] \\
&= 2[\![\alpha(0)\partial_n v_n]\!] - [\![\theta]\!] + [\![\rho]\!]\gamma_a h + \sigma\Delta' h + H_{v,n}(v,h),
\end{aligned}$$

with

$$H_{v,n}(v,h) = -2[\![\alpha(0)\partial_n v_n]\!] + \sqrt{1+|\nabla' h|^2}\big([\![2\alpha(|\mathcal{E}(v,h)|^2)\mathcal{E}(v,h)]\!]\nu\big)_n - \sigma G_\kappa(h).$$

Therefore, we have

$$\begin{array}{rcll}
-[\![\alpha(0)\partial_n v']\!] - [\![\alpha(0)\nabla' v_n]\!] &=& \tilde{H}_v'(v,[\![\theta]\!],h) & \text{on } (0,T_0)\times\mathbb{R}^{n-1}, \\
-2[\![\alpha(0)\partial_n v_n]\!] + [\![\theta]\!] - [\![\rho]\!]\gamma_a h - \sigma\Delta' h &=& H_{v,n}(v,h) & \text{on } (0,T_0)\times\mathbb{R}^{n-1}.
\end{array} \tag{3.15}$$

Taking into account (3.12), the terms $\tilde{H}_v'(v,[\![\theta]\!],h)$ and $H_{v,n}(v,h)$ simplify to

$$\begin{aligned}
\tilde{H}_v'(v,[\![\theta]\!],h) &= -[\![\alpha(0)\partial_n v']\!] - [\![\alpha(0)\nabla' v_n]\!] + \sqrt{1+|\nabla' h|^2}\big([\![2\alpha(|\mathcal{E}(v,h)|^2)\mathcal{E}(v,h)]\!]\nu\big)' \\
&\quad + \sqrt{1+|\nabla' h|^2}\big(-[\![\theta]\!]\nu' + [\![\rho\gamma_a h]\!]\nu' + \sigma\kappa\nu'\big) \\
&= \Big[\!\!\Big[\big(\alpha(|\mathcal{E}(v,h)|^2) - \alpha(0)\big)\big(\partial_n v' + \nabla' v_n\big)\Big]\!\!\Big] \\
&\quad - \Big[\!\!\Big[\alpha(|\mathcal{E}(v,h)|^2)\big((\nabla' v' + (\nabla' v')^T)\nabla' h - |\nabla' h|^2\partial_n v' - (\partial_n v'\cdot\nabla' h)\nabla' h + \partial_n v_n\nabla' h\big)\Big]\!\!\Big] \\
&\quad + [\![\theta]\!]\nabla' h - [\![\rho]\!]\gamma_a h\nabla' h - \sigma\big(\Delta' h - G_\kappa(h)\big)\nabla' h,
\end{aligned}$$

as well as

$$\begin{aligned}
H_{v,n}(v,h) &= -2[\![\alpha(0)\partial_n v_n]\!] + \sqrt{1+|\nabla' h|^2}\big([\![2\alpha(|\mathcal{E}(v,h)|^2)\mathcal{E}(v,h)]\!]\nu\big)_n - \sigma G_\kappa(h) \\
&= 2\Big[\!\!\Big[\big(\alpha(|\mathcal{E}(v,h)|^2) - \alpha(0)\big)\partial_n v_n\Big]\!\!\Big] \\
&\quad - \Big[\!\!\Big[\alpha(|\mathcal{E}(v,h)|^2)\big(\nabla' v_n\cdot\nabla' h + \partial_n v'\cdot\nabla' h - \partial_n v_n|\nabla' h|^2\big)\Big]\!\!\Big] - \sigma G_\kappa(h).
\end{aligned} \tag{3.16}$$

By (3.15), the pressure jump is given by $[\![\theta]\!] = H_{v,n}(v,h) + 2[\![\alpha(0)\partial_n v_n]\!] + [\![\rho]\!]\gamma_a h + \sigma\Delta' h$. Inserting this into $\tilde{H}'_v(v,[\![\theta]\!],h)$, we infer

$$
\begin{aligned}
&H'_v(v,h)\\
&\quad = \tilde{H}_v(v,[\![\theta]\!],h)\\
&\quad = [\![\big(\alpha(|\mathcal{E}(v,h)|^2) - \alpha(0)\big)\big(\partial_n v' + \nabla' v_n\big)]\!]\\
&\qquad - [\![\alpha(|\mathcal{E}(v,h)|^2)\big((\nabla' v' + (\nabla' v')^T)\nabla' h - |\nabla' h|^2 \partial_n v' - (\partial_n v' \cdot \nabla' h)\nabla' h + \partial_n v_n \nabla' h\big)]\!]\\
&\qquad + [\![\theta]\!]\nabla' h - [\![\rho]\!]\gamma_a h \nabla' h - \sigma\big(\Delta' h - G_\kappa(h)\big)\nabla' h\\
&\quad = [\![\big(\alpha(|\mathcal{E}(v,h)|^2) - \alpha(0)\big)\big(\partial_n v' + \nabla' v_n\big)]\!]\\
&\qquad - [\![\alpha(|\mathcal{E}(v,h)|^2)\big((\nabla' v' + (\nabla' v')^T)\nabla' h - |\nabla' h|^2 \partial_n v' - (\partial_n v' \cdot \nabla' h)\nabla' h + \partial_n v_n \nabla' h\big)]\!]\\
&\qquad + H_{v,n}(v,h)\nabla' h + 2[\![\alpha(0)\partial_n v_n]\!]\nabla' h + \sigma G_\kappa(h)\nabla' h.
\end{aligned}
$$

Taking into account (3.16), we deduce that

$$
\begin{aligned}
&(3.17)\\
&\;H'_v(v,h)\\
&\quad = [\![\big(\alpha(|\mathcal{E}(v,h)|^2) - \alpha(0)\big)\big(\partial_n v' + \nabla' v_n\big)]\!]\\
&\qquad - [\![\alpha(|\mathcal{E}(v,h)|^2)\big((\nabla' v' + (\nabla' v')^T)\nabla' h - |\nabla' h|^2 \partial_n v' - (\partial_n v' \cdot \nabla' h)\nabla' h + \partial_n v_n \nabla' h\big)]\!]\\
&\qquad + 2[\![\alpha(0)\partial_n v_n]\!]\nabla' h + 2[\![\big(\alpha(|\mathcal{E}(v,h)|^2) - \alpha(0)\big)\partial_n v_n]\!]\nabla' h\\
&\qquad - [\![\alpha(|\mathcal{E}(v,h)|^2)\big(\nabla' v_n \cdot \nabla' h + \partial_n v' \cdot \nabla' h - \partial_n v_n |\nabla' h|^2\big)]\!]\nabla' h\\
&\quad = [\![\big(\alpha(|\mathcal{E}(v,h)|^2) - \alpha(0)\big)\big(\partial_n v' + \nabla' v_n + 2\partial_n v_n \nabla' h\big)]\!] + 2[\![\alpha(0)\partial_n v_n]\!]\nabla' h\\
&\qquad - [\![\alpha(|\mathcal{E}(v,h)|^2)\times\\
&\qquad\quad \times \big((\nabla' v' + (\nabla' v')^T)\nabla' h - |\nabla' h|^2 \partial_n v' + \partial_n v_n \nabla' h + (\nabla' v_n \cdot \nabla' h)\nabla' h - \partial_n v_n |\nabla' h|^2 \nabla' h\big)]\!].
\end{aligned}
$$

Next, we transform the kinematic condition

$$V = u \cdot \nu \quad \text{on } \Gamma(t).$$

Let $(x'(t), h(t,x'(t)))$ be a point on the hypersurface. The normal velocity of this point is given by

$$
\begin{aligned}
V &= \frac{1}{\sqrt{1+|\nabla' h|^2}}\left(\partial_t \begin{pmatrix} x'(t) \\ h(t,x'(t)) \end{pmatrix}\right) \cdot \begin{pmatrix} -\nabla' h(t,x'(t)) \\ 1 \end{pmatrix}\\
&= \frac{1}{\sqrt{1+|\nabla' h|^2}} \begin{pmatrix} v'(t,x'(t)) \\ (\partial_t h)(t,x'(t)) + v'(t,x'(t)) \cdot \nabla' h(t,x'(t)) \end{pmatrix} \cdot \begin{pmatrix} -\nabla' h(t,x'(t)) \\ 1 \end{pmatrix}\\
&= \frac{1}{\sqrt{1+|\nabla' h|^2}} (\partial_t h)(t,x'(t)).
\end{aligned}
$$

Hence, the kinematic condition transforms to

$$\partial_t h - v_n = H_h(v,h) \quad \text{on } (0,T_0) \times \mathbb{R}^{n-1}, \quad \text{with} \quad H_h(v,h) = -v \cdot \nabla h. \tag{3.18}$$

Finally, we transform the initial values and their compatibility conditions. By assumption, we have $h_0 \in W_p^{3-\frac{2}{p}}(\mathbb{R}^{n-1})$ with $\mathrm{graph}(h_0) = \Gamma_0$ and $v_0 = u_0 \circ \Theta_{h_0}$ with $\Theta_{h_0}(x', x_n) = (x', h_0(x') + x_n)$.

The transformed compatibility conditions are given by

$$\operatorname{div} v_0 = F_d(v_0, h_0) \quad \text{in } \dot{\mathbb{R}}^n, \quad [\![v_0]\!] = 0 \quad \text{on } \mathbb{R}^{n-1}, \tag{3.19}$$

$$\text{and} \quad \left([\![\alpha(|\mathcal{E}(v_0, h_0)|^2)\mathcal{E}(v_0, h_0)]\!]\nu_0\right)_{\tan} = 0 \quad \text{on } \mathbb{R}^{n-1}. \tag{3.20}$$

We construct a suitable equivalent formulation of the compatibility condition (3.20). The compatibility condition

$$\left([\![\alpha(|Eu_0|^2)Eu_0]\!]\nu_0\right)_{\tan} = 0 \quad \text{on } \Gamma_0$$

in combination with the definition the initial value of the pressure jump (3.3)

$$[\![\pi_0]\!] = 2[\![\alpha(|Eu_0|^2)(Eu_0\nu_0) \cdot \nu_0]\!] + \sigma\kappa \quad \text{on } \Gamma_0.$$

are equivalent to

$$-[\![\alpha(|Eu_0|^2)Eu_0 - \pi_0]\!]\nu_0 = \sigma\kappa\nu_0 \quad \text{on } \Gamma_0. \tag{3.21}$$

We define the initial value for the transforemd pressure jump $[\![\theta_0]\!] := [\![\pi_0 \circ \Theta_{h_0}]\!] + [\![\rho\gamma_a h_0]\!]$ according to the transformed pressure $\theta = \pi_1 \circ \Theta_h$ (for the definition of π_1 see (3.4)). The boundary condition (3.21) transforms to

$$-[\![2\alpha(|\mathcal{E}(v_0, h_0)|^2)\mathcal{E}(v_0, h_0) - \theta_0 + \rho\gamma_a h_0]\!]\nu_0 = \sigma(\Delta' h_0 - G_\kappa(h_0))\nu_0 \quad \text{on } \mathbb{R}^{n-1}.$$

By the definition of H_v, this equation is equivalent to (compare to (3.15))

$$\begin{aligned} -[\![\alpha(0)\partial_n v_0']\!] - [\![\alpha(0)\nabla' v_{0,n}]\!] &= H_v'(v_0, h_0) && \text{on } \mathbb{R}^{n-1}, \\ -2[\![\alpha(0)\partial_n v_{0,n}]\!] + [\![\theta_0]\!] - [\![\rho]\!]\gamma_a h_0 - \sigma\Delta' h_0 &= H_{v,n}(v_0, h_0) && \text{on } \mathbb{R}^{n-1}. \end{aligned}$$

Hence, we can replace the compatibility condition (3.20) by

$$-[\![\alpha(0)\partial_n v_0']\!] - [\![\alpha(0)\nabla' v_{0,n}]\!] = H_v'(v_0, h_0) \quad \text{on } \mathbb{R}^{n-1}. \tag{3.22}$$

In summary, the transformed system reads:

$$\left\{ \begin{aligned} \rho\partial_t v - \alpha(0)\Delta v + \nabla\theta &= F(v, \theta, h) && \text{in } (0, T_0) \times \dot{\mathbb{R}}^n, \\ \operatorname{div} v &= F_d(v, h) && \text{in } (0, T_0) \times \dot{\mathbb{R}}^n, \\ [\![v]\!] &= 0 && \text{on } (0, T_0) \times \mathbb{R}^{n-1}, \\ -[\![\alpha(0)\partial_n v']\!] - [\![\alpha(0)\nabla' v_n]\!] &= H_v'(v, h) && \text{on } (0, T_0) \times \mathbb{R}^{n-1}, \\ -2[\![\alpha(0)\partial_n v_n]\!] + [\![\theta]\!] - [\![\rho]\!]\gamma_a h - \sigma\Delta' h &= H_{v,n}(v, h) && \text{on } (0, T_0) \times \mathbb{R}^{n-1}, \\ \partial_t h - v_n &= H_h(v, h) && \text{on } (0, T_0) \times \mathbb{R}^{n-1}, \\ v(0) &= v_0 && \text{in } \dot{\mathbb{R}}^n, \\ h(0) &= h_0 && \text{on } \mathbb{R}^{n-1}, \end{aligned} \right. \tag{3.23}$$

with the nonlinearities (see (3.10), (3.11), and (3.16)– (3.18))

$$
\begin{aligned}
F(v,\theta,h)_j &= \rho\partial_n v_j\partial_t h - \rho\big(v.\nabla v_j - (v'\cdot\nabla' h)\partial_n v_j\big) + \partial_n\theta\partial_j h \\
&\quad - \alpha(0)\big(2(\partial_n\nabla' v_j)\cdot\nabla' h - \partial_n^2 v_j|\nabla' h|^2 + \partial_n v_j\Delta' h\big) \\
&\quad + \sum_{k,l,m}\big(\mathcal{A}^{l,m}_{j,k}(\mathcal{E}(v,h)) - \mathcal{A}^{l,m}_{j,k}(0)\big)\times \\
&\qquad \times\big(\partial_l\partial_m v_k - \partial_m\partial_n v_k\partial_l h - \partial_l\partial_n v_k\partial_m h + \partial_n^2 v_k\partial_l h\partial_m h - \partial_n v_k\partial_l\partial_m h\big), \quad j=1,\dots,n, \\
F_d(v,h) &= (\partial_n v)\cdot\nabla h, \\
H_v'(v,h) &= \big[\!\big[\big(\alpha(|\mathcal{E}(v,h)|^2)-\alpha(0)\big)\big(\partial_n v' + \nabla' v_n + 2\partial_n v_n\nabla' h\big)\big]\!\big] + 2[\![\alpha(0)\partial_n v_n]\!]\nabla' h \\
&\quad - [\![\alpha(|\mathcal{E}(v,h)|^2)\times \\
&\qquad \times\big((\nabla' v' + (\nabla' v')^T)\nabla' h - |\nabla' h|^2\partial_n v' + \partial_n v_n\nabla' h + (\nabla' v_n\cdot\nabla' h)\nabla' h \\
&\qquad\quad - \partial_n v_n|\nabla' h|^2\nabla' h\big)\big]\!\big], \\
H_{v,n}(v,h) &= 2\big[\!\big[\big(\alpha(|\mathcal{E}(v,h)|^2)-\alpha(0)\big)\partial_n v_n\big]\!\big] - [\![\alpha(|\mathcal{E}(v,h)|^2)\big(\nabla' v_n\cdot\nabla' h + \partial_n v'\cdot\nabla' h - \partial_n v_n|\nabla' h|^2\big) \\
&\quad - \sigma G_\kappa(h), \\
H_h(v,h) &= -v\cdot\nabla h,
\end{aligned}
$$

and the transformed initial value $v_0(x',x_n) = u_0(x', x_n + h_0(x',x_n))$, $(x',x_n)\in\mathbb{R}^n$, where the function $h_0 \in W_p^{3-\frac{2}{p}}(\mathbb{R}^{n-1})$ with $\operatorname{graph} h_0 = \Gamma_0$ is given by assumption. Furthermore, we have the transformed compatibility conditions (see (3.19) and (3.22))

$$
\begin{aligned}
&\operatorname{div} v_0 = F_d(v_0,h_0) \quad \text{in } \dot{\mathbb{R}}^n \quad \text{and} \\
&[\![v_0]\!] = 0, \quad -[\![\alpha(0)\partial_n v_0']\!] - [\![\alpha(0)\nabla' v_{0,n}]\!] = H_v'(v_0,h_0) \quad \text{on } \mathbb{R}^{n-1}.
\end{aligned} \tag{3.24}
$$

In order to shorten notation, we introduce the nonlinear term

$$
N(v,\theta,h) = (F(v,\theta,h), F_d(v,h), H_v(v,h), H_h(v,h)).
$$

Fixed point formulation

We rewrite (3.23) in form of a fixed point problem in a suitable metric space. Let $n+2<p<\infty$. We define the solution spaces with and without the prescribed initial values by

$$
\begin{aligned}
\mathbb{E} &:= \{(v,\theta,h)\in\mathbb{E}_u(T_0,\dot{\mathbb{R}}^n)\times\mathbb{E}_\pi(T_0,\dot{\mathbb{R}}^n)\times\mathbb{E}_h(T_0,\mathbb{R}^{n-1})\colon [\![v]\!]=0 \text{ on } \mathbb{R}^{n-1}\}, \\
{}_{v_0,h_0}\mathbb{E} &:= \big\{(v,\theta,h)\in\mathbb{E}\colon v(0)=v_0,\ h(0)=h_0\big\},
\end{aligned}
$$

with

$$
\begin{aligned}
\mathbb{E}_u(T_0,\dot{\mathbb{R}}^n) &:= H^1_p(0,T_0;L_p(\mathbb{R}^n))\cap L_p(0,T;H^2_p(\dot{\mathbb{R}}^n)), \\
\mathbb{E}_\pi(T_0,\dot{\mathbb{R}}^n) &:= L_p(0,T_0;\widehat{H}^1_p(\dot{\mathbb{R}}^n)), \\
\mathbb{E}_h(T_0,\mathbb{R}^{n-1}) &= W_p^{2-\frac{1}{2p}}(0,T_0;L_p(\mathbb{R}^{n-1}))\cap H^1_p(0,T_0;W_p^{2-\frac{1}{p}}(\mathbb{R}^{n-1}))\cap L_p(0,T_0;W_p^{3-\frac{1}{p}}(\mathbb{R}^{n-1})),
\end{aligned}
$$

and the space for the data, with and without prescribed compatibility conditions by

$$
\begin{aligned}
\mathbb{F} &:= \mathbb{F}_f(T_0,\mathbb{R}^n)\times\mathbb{F}_d(T_0,\mathbb{R}^n,\dot{\mathbb{R}}^n)\times\mathbb{H}_u(T_0,\mathbb{R}^{n-1})\times\mathbb{H}_h(T_0,\mathbb{R}^{n-1}), \\
\mathbb{F}_c &:= \big\{(f,f_d,h_u,h_h)\in\mathbb{F}\colon \\
&\qquad f(0)=\operatorname{div} v_0 \text{ in } \dot{\mathbb{R}}^n,\ -[\![\alpha(0)\partial_n v_0']\!] - [\![\alpha(0)\nabla' v_{0,n}]\!] = h_u'(0) \text{ on } \mathbb{R}^{n-1}\big\},
\end{aligned}
$$

with

$$
\begin{array}{lll}
\mathbb{F}_f(T_0, \mathbb{R}^n) & := & L_p(0, T_0; L_p(\mathbb{R}^n)), \\
\mathbb{F}_d(T_0, \mathbb{R}^n, \dot{\mathbb{R}}^n) & := & H^1_p(0, T_0; \widehat{H}^{-1}_p(\mathbb{R}^n)) \cap L_p(0, T_0; H^1_p(\dot{\mathbb{R}}^n)), \\
\mathbb{H}_u(T_0, \mathbb{R}^{n-1}) & = & \big\{ h_u = (h'_u, h_n) \in W_p^{\frac{1}{2} - \frac{1}{2p}}(0, T_0; L_p(\mathbb{R}^{n-1})) \cap L_p(0, T_0; W_p^{1-\frac{1}{p}}(\mathbb{R}^{n-1})) \big\}, \\
\mathbb{H}_h(T_0, \mathbb{R}^{n-1}) & = & W_p^{1-\frac{1}{2p}}(0, T_0; L_p(\mathbb{R}^{n-1})) \cap L_p(0, T_0; W_p^{2-\frac{1}{p}}(\mathbb{R}^{n-1})).
\end{array}
$$

The spaces $\mathbb{E}_h(T_0, \mathbb{R}^{n-1})$, $\mathbb{H}_u(T_0, \mathbb{R}^{n-1})$, and $\mathbb{H}_h(T_0, \mathbb{R}^{n-1})$ are already defined in the preliminaries. If we replace $\dot{\mathbb{R}}^n$ by a domain Ω in the definition of $\mathbb{E}_u(T_0, \dot{\mathbb{R}}^n)$, we have the space $\mathbb{E}_u(T_0, \Omega)$, which was also defined in the preliminaries.

Problem (3.23) can be written as a fixed point problem of the map

$$
\Phi \colon {}_{v_0,h_0}\mathbb{E} \to {}_{v_0,h_0}\mathbb{E}, \quad (v, \theta, h) \mapsto \tilde{\Phi}_{v_0,h_0}(N(v, \theta, h)),
$$

where

$$
\tilde{\Phi}_{v_0,h_0} \colon \mathbb{F}_c \to {}_{v_0,h_0}\mathbb{E}, \quad (\tilde{f}, \tilde{f}_d, \tilde{h}_u, \tilde{h}_h) \to (v, \theta, h)
$$

denotes the solution operator to the following problem:

$$
\left\{
\begin{array}{rcll}
\rho \partial_t v - \alpha(0) \Delta v + \nabla \theta & = & \tilde{f} & \text{in } (0, T_0) \times \dot{\mathbb{R}}^n, \\
\operatorname{div} v & = & \tilde{f}_d & \text{in } (0, T_0) \times \dot{\mathbb{R}}^n, \\
[\![v]\!] & = & 0 & \text{on } (0, T_0) \times \mathbb{R}^{n-1}, \\
-[\![\alpha(0) \partial_n v']\!] - [\![\alpha(0) \nabla' v_n]\!] & = & \tilde{h}'_u & \text{on } (0, T_0) \times \mathbb{R}^{n-1}, \\
-2[\![\alpha(0) \partial_n v_n]\!] + [\![\theta]\!] - [\![\rho]\!] \gamma_a h - \sigma \Delta' h & = & \tilde{h}_{u,n} & \text{on } (0, T_0) \times \mathbb{R}^{n-1}, \\
\partial_t h - v_n & = & \tilde{h}_h & \text{on } (0, T_0) \times \mathbb{R}^{n-1}, \\
v(0) & = & v_0 & \text{in } \dot{\mathbb{R}}^n, \\
h(0) & = & h_0 & \text{on } \mathbb{R}^{n-1}.
\end{array}
\right.
$$

The compatibility conditions in the proposition on the solvability of the linearized problem (Proposition 1.9) are fulfilled, due to the definition of $\mathbb{F}_c$. Thus $\tilde{\Phi}_{v_0,h_0}$ is well defined. In Lemma 3.2, we show $N(v, \theta, h) \in \mathbb{F}_c$, provided that $(v, \theta, h) \in {}_{v_0,h_0}\mathbb{E}$. Hence, Φ is well-defined.

Mapping properties of the nonlinearities

Our next subject is the mapping properties of the nonlinearity N. In the next two lemmas, we show that $N \colon \mathbb{E} \to \mathbb{F}$ is well defined and continuously Fréchet differentiable and $\mathbb{N}({}_{v_0,h_0}\mathbb{E}) \subset \mathbb{F}_c$. This forms the basis of the fixed point argument.

We recall the definition of the auxiliary space $\mathbb{H}_u^\infty$ for the term $\alpha(|\mathcal{E}(v,h)|^2)$ appearing in H_v on the boundary. We will be able to show that $\mathcal{E}(v, h) \in \mathbb{H}_u(T_0, \mathbb{R}^{n-1})$, but in general $\alpha(|\mathcal{E}(v,h)|^2)$ only belongs to the larger space $\mathbb{H}_u^\infty$ (see Proposition 1.17)

$$
\begin{aligned}
\mathbb{H}_u^\infty(T_0, \mathbb{R}^{n-1}) = \{ u \in BUC([0, T_0], BUC(\mathbb{R}^{n-1})) \colon & \\
\|u\|_{\mathbb{H}_u^\infty(T_0, \mathbb{R}^{n-1})} = \|u\|_{T_0, \mathbb{R}^{n-1}, \infty, \infty} + [u]_{\mathbb{H}_u(T_0, \mathbb{R}^{n-1})} < \infty \}.
\end{aligned}
$$

We recall the definition

$$
\begin{aligned}
&[u]_{\mathbb{H}_u(T_0,\mathbb{R}^{n-1})}\\
&\quad=\Big(\int_0^{T_0}\int_0^{T_0}\frac{\|u(t)-u(s)\|^p_{p,\Gamma}}{|t-s|^{\frac12+\frac p2}}dsdt\Big)^{\frac1p}+\Big(\int_0^{T_0}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}}\frac{|u(t,x)-u(t,y)|^p}{|x-y|^{n-2+p}}dxdydt\Big)^{\frac1p}\\
&\quad=\Big(\int_{\mathbb{R}^{n-1}}[u(\cdot,x)]^p_{W_p^{\frac12-\frac1{2p}}(0,T_0)}dx\Big)^{\frac1p}+\Big(\int_0^{T_0}[u(t,\cdot)]_{W_p^{1-\frac1p}(\mathbb{R}^{n-1})}dt\Big)^{\frac1p}.
\end{aligned}
$$

The space $\mathbb{H}_u^\infty(T_0,\mathbb{R}^{n-1})$ is defined and discussed in the preliminaries (see Section 1.4).

Lemma 3.2. *Fix $n\in\mathbb{N}$, $n\geq 2$, $n+2<p<\infty$, and $T_0,\rho_\pm,\sigma>0$. Assume that $\alpha_\pm\in C^3([0,\infty))$ and $(v_0,h_0)\in W_p^{2-\frac2p}(\dot{\mathbb{R}}^n)\times W_p^{3-\frac1p}(\mathbb{R}^{n-1})$ satisfy the compatibility conditions* (3.24)*. Then*

$$
\begin{array}{rcll}
N(v,\theta,h) & \in & \mathbb{F} & (v,\theta,h)\in\mathbb{E}\\
N(v,\theta,h) & \in & \mathbb{F}_c & (v,\theta,h)\in{}_{v_0,h_0}\mathbb{E}.
\end{array}
$$

Proof. First, we prove that $F(v,\theta,h)\in\mathbb{F}_f(T_0,\mathbb{R}^n)$. By the proposition on embedding theorems (Proposition 1.14), we obtain

$$
\nabla v\in BUC([0,T_0],BUC(\dot{\mathbb{R}}^n))\quad\text{and}\quad\nabla' h\in BUC([0,T_0],BUC(\mathbb{R}^{n-1})).
$$

Hence, it is

$$
\mathcal{E}(v,h)\in BUC([0,T_0],BUC(\dot{\mathbb{R}}^n)), \tag{3.25}
$$

and therefore $\mathcal{A}^{l,m}_{j,k}(\mathcal{E}(v,h))\in BUC([0,T_0],BUC(\dot{\mathbb{R}}^n))$. We recall the definition

$$
\begin{aligned}
F(v,\theta,h)_j&=\rho\partial_n v_j\partial_t h-\rho\big(v.\nabla v_j-(v'\cdot\nabla' h)\partial_n v_j\big)+\partial_n\theta\partial_j h\\
&\quad-\alpha(0)\big(2(\partial_n\nabla' v_j)\cdot\nabla' h-\partial_n^2 v_j|\nabla' h|^2+\partial_n v_j\Delta' h\big)\\
&\quad+\sum_{k,l,m}\big(\mathcal{A}^{l,m}_{j,k}(\mathcal{E}(v,h))-\mathcal{A}^{l,m}_{j,k}(0)\big)\times\\
&\quad\quad\times\big(\partial_l\partial_m v_k-\partial_m\partial_n v_k\partial_l h-\partial_l\partial_n v_k\partial_m h+\partial_n^2 v_k\partial_l h\partial_m h-\partial_n v_k\partial_l\partial_m h\big),\quad j=1,\dots,n.
\end{aligned}
$$

We emphasize that $F(v,\theta,h)$ consists of sums and products of the terms (see the proposition on embedding theorems)

$$
v,\partial_t h,\nabla' h,\nabla'^2 h,\rho,\alpha(0),\big(\mathcal{A}^{l,m}_{j,k}(0)\big)^n_{j,k,l,m=1},\big(\mathcal{A}^{l,m}_{j,k}(\mathcal{E}(v,h))\big)^n_{j,k,l,m=1}\in BUC([0,T_0],BUC(\dot{\mathbb{R}}^n)),
$$

and

$$
\nabla^2 v,\nabla\theta\in\mathbb{F}_f(T_0,\mathbb{R}^n),
$$

and

$$
\nabla v\in BUC([0,T_0],BUC(\dot{\mathbb{R}}^n))\cap\mathbb{F}_f(T_0,\mathbb{R}^n).
$$

Each summand of $F(v,\theta,h)$ is the product of one and only one of the terms listed above in $\mathbb{F}_f(T_0,\mathbb{R}^n)$ and at least one term of the terms listed above in $BUC([0,T_0],BUC(\dot{\mathbb{R}}^n))$. This implies

$$F(v,\theta,h)\in\mathbb{F}_f(T_0,\mathbb{R}^n).$$

Second, we prove $F_d(v,h)\in\mathbb{F}_d(T_0,\mathbb{R}^n,\dot{\mathbb{R}}^n)$. We recall the definition $F_d(v,h)=(\partial_n v)\cdot\nabla h$. Since

$$\nabla h\in BUC([0,T_0],BUC^1(\mathbb{R}^{n-1}))\quad\text{and}\quad \partial_n v\in L_p(0,T_0;H^1_p(\dot{\mathbb{R}}^n)),$$

by the proposition on embedding theorems and the definition of $\mathbb{E}_u(T_0,\dot{\mathbb{R}}^n)$, it follows that

$$F_d(v,h)\in L_p(0,T_0;H^1_p(\dot{\mathbb{R}}^n)).$$

Since h does not depend on x_n, it holds

$$F_d(v,h)=(\partial_n v)\cdot\nabla h=\partial_n(v\cdot\nabla h).$$

By the proposition on embedding theorems and the definition of $\mathbb{E}_u(T_0,\dot{\mathbb{R}}^n)$, we have

$$\nabla h\in BUC^1([0,T_0],BUC(\mathbb{R}^{n-1}))\quad\text{and}\quad v\in H^1_p(0,T_0;L_p(\mathbb{R}^n)),$$

and hence $v\cdot\nabla h\in H^1_p(0,T_0;L_p(\mathbb{R}^n))$. The continuity of the operator

$$\partial_n\colon H^1_p(0,T_0;L_p(\mathbb{R}^n))\to H^1_p(0,T_0;\widehat{H}^{-1}_p(\mathbb{R}^n))$$

establishes $F_d(v,h)\in H^1_p(0,T_0;\widehat{H}^{-1}_p(\mathbb{R}^n))$. In summary, we have $F_d(v,h)\in\mathbb{F}_d(T_0,\mathbb{R}^n,\dot{\mathbb{R}}^n)$.

Third, we show that $H_v(v,h)\in\mathbb{H}_u(T_0,\mathbb{R}^{n-1})$. For this purpose, we prove that

$$\nabla'^2 h\in\mathbb{H}_u(T_0,\mathbb{R}^{n-1}).\tag{3.26}$$

By the definition of $\mathbb{E}_h(T_0,\mathbb{R}^{n-1})$, it follows that

$$\nabla'^2 h\in L_p(0,T_0;W_p^{1-\frac{1}{p}}(\mathbb{R}^{n-1})).$$

The mixed derivative theorem (see Denk, Saal, and Seiler [DSS08, Lemma 4.3]) implies

$$\begin{aligned}\mathbb{E}_h(T_0,\mathbb{R}^{n-1})&\hookrightarrow H^1_p(0,T_0;W_p^{2-\frac{1}{p}}(\mathbb{R}^{n-1}))\cap L_p(0,T_0;W_p^{3-\frac{1}{p}}(\mathbb{R}^{n-1}))\\&\hookrightarrow H^\alpha_p(0,T_0;W_p^{3-\alpha-\frac{1}{p}}(\mathbb{R}^{n-1})),\quad \alpha\in(0,1)\setminus\{1-\frac{1}{p}\}.\end{aligned}$$

Choosing now $\frac{1}{2}-\frac{1}{2p}<\alpha<1-\frac{1}{p}$ and using the embedding $W_p^{3-\alpha-\frac{1}{p}}(\mathbb{R}^{n-1})\hookrightarrow H^2_p(\mathbb{R}^{n-1})$, it follows that $\nabla'^2 h\in\mathbb{H}_u(T_0;\mathbb{R}^{n-1})$. Next, we show that $G_\kappa(h)\in\mathbb{H}_u(T_0;\mathbb{R}^{n-1})$. We use the special form of $G_\kappa(h)$ (see (3.13) for the definition of G_κ), i.e.

$$G_\kappa(h)=\Psi_1(|\nabla' h|^2)\Delta' h+\Psi_2(|\nabla' h|^2)\nabla' h\cdot(\nabla'^2 h\nabla' h),$$

with

$$\Psi_1(x) = \Big(1 - \frac{1}{\sqrt{1+x}}\Big) \quad \text{and} \quad \Psi_2(x) = \frac{1}{(1+x)^{\frac{3}{2}}}, \quad x \geq 0.$$

We choose a smooth extension for $x < 0$, such that $\Psi_1, \Psi_2 \in C^\infty(\mathbb{R})$. By the proposition on pointwise multiplications (Proposition 1.16), we deduce that $|\nabla h|^2 \in \mathbb{H}_u(T_0; \mathbb{R}^{n-1})$. Hence,

$$\Psi_1(|\nabla h|^2), \Psi_2(|\nabla h|^2) \in \mathbb{H}_u^\infty(T_0; \mathbb{R}^{n-1}), \tag{3.27}$$

by the proposition on Nemytskij operators (Proposition 1.17). Applying once more the proposition on pointwise multiplications and taking into account (3.26) and (3.27), we deduce that

$$G_\kappa(h) \in \mathbb{H}_u(T_0; \mathbb{R}^{n-1}). \tag{3.28}$$

By the proposition on trace theorems (Proposition 1.15), it holds

$$\gamma_\pm \nabla v \in \mathbb{H}_u(T_0; \mathbb{R}^{n-1}),$$

and since $\nabla' h \in \mathbb{H}_u(T_0; \mathbb{R}^{n-1})$, we obtain with proposition on pointwise multiplications also

$$\gamma_\pm \mathcal{E}(v, h) \in \mathbb{H}_u(T_0; \mathbb{R}^{n-1}). \tag{3.29}$$

The proposition on Nemytskij operators yields

$$\alpha(\gamma_\pm |\mathcal{E}(v, h)|^2) \in \mathbb{H}_u^\infty(T_0; \mathbb{R}^{n-1}).$$

We recall the definition of H_v

$$\begin{aligned}
H_v'(v,h) = {} & \big[\!\big[\big(\alpha(|\mathcal{E}(v,h)|^2) - \alpha(0)\big)\big(\partial_n v' + \nabla' v_n + 2\partial_n v_n \nabla' h\big) \big]\!\big] + 2[\![\alpha(0)\partial_n v_n]\!]\nabla' h \\
& - \big[\!\big[\alpha(|\mathcal{E}(v,h)|^2) \times \\
& \quad \times \big((\nabla' v' + (\nabla' v')^T)\nabla' h - |\nabla' h|^2 \partial_n v' + \partial_n v_n \nabla' h + (\nabla' v_n \cdot \nabla' h)\nabla' h \\
& \qquad - \partial_n v_n |\nabla' h|^2 \nabla' h \big) \big]\!\big], \\
H_{v,n}(v,h) = {} & 2\big[\!\big[\big(\alpha(|\mathcal{E}(v,h)|^2) - \alpha(0)\big)\partial_n v_n \big]\!\big] - \big[\!\big[\alpha(|\mathcal{E}(v,h)|^2)\big(\nabla' v_n \cdot \nabla' h + \partial_n v' \cdot \nabla' h - \partial_n v_n |\nabla' h|^2\big) \\
& - \sigma G_\kappa(h),
\end{aligned}$$

The nonlinearity H_v is a sum of products of the terms

$$\alpha_\pm(0), \alpha_\pm(|\gamma_\pm \mathcal{E}(v,h)|^2) \in \mathbb{H}_u^\infty(T_0; \mathbb{R}^{n-1}) \quad \text{and} \quad \gamma_\pm \nabla v, \nabla' h, \sigma G_\kappa(h) \in \mathbb{H}_u(T_0; \mathbb{R}^{n-1}),$$

where each summand at least contains one term in $\mathbb{H}_u(T_0; \mathbb{R}^{n-1})$. By the proposition on pointwise multiplications, it follows that $H_v(v,h) \in \mathbb{H}_u(T_0; \mathbb{R}^{n-1})$.

Fourth, we show that $H_h(v,h) \in \mathbb{H}_h(T_0; \mathbb{R}^{n-1})$. Note first that $v \cdot \nabla' h$ is well-defined, since $[\![v]\!] = 0$. We have

$$\nabla' h \in H_p^1(0, T_0; W_p^{1-\frac{1}{p}}(\mathbb{R}^{n-1})) \cap L_p(0, T_0; W_p^{2-\frac{1}{p}}(\mathbb{R}^{n-1})) \hookrightarrow \mathbb{H}_h(T_0, \mathbb{R}^{n-1}).$$

By the proposition on trace theorems, it holds $\gamma v \in \mathbb{H}_h(T_0, \mathbb{R}^{n-1})$. Hence, the proposition on pointwise multiplications implies

$$H_h(v, h) = v \cdot \nabla' h \in \mathbb{H}_h(T_0, \mathbb{R}^{n-1}).$$

This proves $N(v, \theta, h) \in \mathbb{F}$. We now assume that $v(0) = v_0$ and $h(0) = h_0$. The conditions

$$H'_v(v_0, h_0) = -[\![\alpha(0)\partial_n v'_0]\!] - [\![\alpha(0)\nabla' v_{0,n}]\!] \quad \text{on } \mathbb{R}^{n-1} \quad \text{and} \quad \operatorname{div} v_0 = F_d(v_0, h_0) \quad \text{in } \dot{\mathbb{R}}^n$$

are two of the transformed compatibility conditions (3.24). Hence $N(v, \theta, h) \in \mathbb{F}_c$. □

Lemma 3.3. *Fix $n \in \mathbb{N}$, $n \geq 2$, $n + 2 < p < \infty$, and $T_0, \rho_\pm, \sigma > 0$. Then, for $\alpha_\pm \in C^3([0, \infty))$, it is*

$$N \in C^1(\mathbb{E}, \mathbb{F}), \quad N(0) = 0, \quad \textit{and} \quad DN(0) = 0.$$

Proof. It is $N \colon \mathbb{E} \to \mathbb{F}$ by Lemma 3.2. The nonlinearity N is polynomial in the terms of the set $X = X_0 \cup X_1$, where

$$X_0 = \Big\{ v, \nabla v, \nabla v^2, \gamma v, \gamma_\pm \nabla v, \partial_n \theta, \partial_t h, \nabla' h, \nabla'^2 h, \big(\alpha_\pm(|\gamma_\pm \mathcal{E}(v, h)|^2) - \alpha_\pm(0)\big), \\ \big(\mathcal{A}^{l,m}_{j,k}(\mathcal{E}(v, h)) - \mathcal{A}^{l,m}_{j,k}(0)\big)^n_{j,k,l,m=1}, \sigma G_\kappa(h) \Big\},$$

and

$$X_1 = \Big\{ \alpha_\pm(|(\mathcal{E}(v, h)|^2) \Big\}.$$

If we show that

$$\alpha_\pm(|\gamma_\pm \mathcal{E}(v, h)|^2), \quad \mathcal{A}^{l,m}_{j,k}(\mathcal{E}(v, h)), \quad \text{and} \quad G_\kappa(h)$$

are continuously Fréchet differentiable, it follows that N is continuously Fréchet differentiable.

First, we investigate $\alpha_\pm(|\gamma_\pm \mathcal{E}(v, h)|^2)$. Since $\gamma_\pm \mathcal{E}(v, h)$ is polynomial in $\gamma_\pm \nabla v$ and $\nabla' h$, the trace operator is linear, and $\gamma_\pm \mathcal{E}(v, h) \in \mathbb{H}_u(T_0, \mathbb{R}^{n-1})$ (see (3.29)), we have

$$\gamma_\pm \mathcal{E} \in C^\infty(\mathbb{E}_u(T_0, \dot{\mathbb{R}}^n) \times \mathbb{E}_h(T_0, \mathbb{R}^{n-1}), \mathbb{H}_u(T_0, \mathbb{R}^{n-1})).$$

The proposition on Nemytskij operators (Proposition 1.17) and the chain rule imply, that the map

$$\mathbb{E}_u(T_0, \dot{\mathbb{R}}^n) \times \mathbb{E}_h(T_0, \mathbb{R}^{n-1}) \to \mathbb{H}^\infty_u(T_0, \mathbb{R}^{n-1}), \quad (v, h) \to \alpha_\pm(\gamma_\pm |\mathcal{E}(v, h)|^2)$$

is continuously Fréchet differentiable.

Next, we analyse $\mathcal{A}^{l,m}_{j,k}(\mathcal{E}(v, h))$. We recall $\mathcal{E}(v, h) \in BUC([0, T_0], BUC(\dot{\mathbb{R}}^n))$ (see (3.25)). Since $\mathcal{E}(v, h)$ is polynomial in ∇v and $\nabla' h$, it holds

$$\mathcal{E} \in C^\infty(\mathbb{E}_u(T_0, \dot{\mathbb{R}}^n) \times \mathbb{E}_h(T_0, \mathbb{R}^{n-1}), BUC([0, T_0], BUC(\dot{\mathbb{R}}^n))).$$

By the proposition on Nemytskij operators (Proposition 1.17), we deduce that

$$\mathbb{E}_u(T_0, \dot{\mathbb{R}}^n) \times \mathbb{E}_h(T_0, \mathbb{R}^{n-1}) \to BUC([0, T_0], BUC(\dot{\mathbb{R}}^n)), \quad (v, h) \to \mathcal{A}^{l,m}_{j,k}(\mathcal{E}(v, h))$$

is continuously Fréchet differentiable.

Next, we investigate the Fréchet differentiability of $G_\kappa(h)$ and show that $G_\kappa(h) = 0$ as well as $DG_\kappa(h) = 0$. We recall $G_\kappa(h) \in \mathbb{H}_u(T_0, \mathbb{R}^{n-1})$ (see (3.28)). Once more, we use the special form of G_κ, i.e.

$$G_\kappa(h) = \Psi_1(|\nabla' h|^2)\Delta' h + \Psi_2(|\nabla' h|^2)(\nabla' h|\nabla'^2 h\nabla' h),$$

with

$$\Psi_1(x) = \Big(1 - \frac{1}{\sqrt{1+x}}\Big) \quad \text{and} \quad \Psi_2(x) = \frac{1}{(1+x)^{\frac{3}{2}}}, \quad x \geq 0.$$

We choose a smooth extension for $x < 0$, such that $\Psi_1, \Psi_2 \in C^\infty(\mathbb{R})$. It holds $G_\kappa(0) = 0$. By the proposition on Nemytskij operators, it follows that

$$\mathbb{H}_u(T_0, \mathbb{R}^{n-1}) \to \mathbb{H}_u^\infty(T_0, \mathbb{R}^{n-1}), \quad |\nabla' h|^2 \mapsto \Psi_j(|\nabla h|^2), \quad j = 1, 2$$

is continuously Fréchet differentiable. Further, the maps

$$\mathbb{E}_h(T_0, \mathbb{R}^{n-1}) \to \mathbb{H}_u(T_0, \mathbb{R}^{n-1}), \quad h \mapsto |\nabla' h|^2,$$

and

$$\mathbb{E}_h(T_0, \mathbb{R}^{n-1}) \to \mathbb{H}_u(T_0, \mathbb{R}^{n-1}), \quad h \mapsto \Delta' h,$$

and

$$\mathbb{E}_h(T_0, \mathbb{R}^{n-1}) \to \mathbb{H}_u(T_0, \mathbb{R}^{n-1}), \quad h \mapsto (\nabla' h|\nabla'^2 h\nabla' h),$$

are smooth, since they are polynomial. By the product rule and the proposition on pointwise multiplications (Proposition 1.16), we infer

$$G_\kappa(h) \in C^1(\mathbb{E}_h(T_0, \mathbb{R}^{n-1}), \mathbb{H}_u(T_0, \mathbb{R}^{n-1})).$$

To compute the Fréchet derivative, we use

$$D(|\nabla' h|^2)[\bar{h}] = 2\nabla h \cdot \nabla \bar{h}, \quad \bar{h} \in \mathbb{H}_u(T_0, \mathbb{R}^{n-1}),$$

and hence

$$\begin{aligned} D(G_\kappa(h))[\bar{h}] = {} & 2\Psi_1'(|\nabla' h|^2)(\nabla' h \cdot \nabla' \bar{h})\Delta' h + \Psi_1(|\nabla' h|^2)\Delta' \bar{h} \\ & + 2\Psi_2'(|\nabla' h|^2)(\nabla' h \cdot \nabla' \bar{h})\nabla' h \cdot (\nabla'^2 h\nabla' h) + \Psi_2(|\nabla' h|^2)\nabla' \bar{h} \cdot (\nabla'^2 h\nabla' h) \\ & + \Psi_2(|\nabla' h|^2)\nabla' h \cdot (\nabla'^2 \bar{h}\nabla' h) + \Psi_2(|\nabla' h|^2)\nabla' h \cdot (\nabla'^2 h\nabla' \bar{h}), \quad \bar{h} \in \mathbb{H}_u(T_0, \mathbb{R}^{n-1}). \end{aligned}$$

Inserting $h = 0$ and using $\Psi_1(0) = 0$, it follows that $DG_\kappa(0) = 0$.

For each element $T(v, \theta, h) \in X_0$, we have $T(0) = 0$. Each summand of N, excluding $\sigma G_\kappa(h)$, is a product of at least two factors of X_0 and at least one factor of X_1. Since $G_\kappa(h) = 0$ and $DG_\kappa(h) = 0$, this implies with the product rule $N(0) = 0$ and $DN(0) = 0$. □

Fixed point argument

We are now in a position to apply the contraction mapping principle in the metric space ${}_{v_0,h_0}\mathbb{E}$. In Lemma 3.3, we proved that $N\colon \mathbb{E} \to \mathbb{F}$ is continuously Fréchet differentiable with $N(0) = 0$ and $DN(0) = 0$. Hence, for $\eta > 0$ we can choose $R > 0$ such that

$$\sup_{\bar{z}\in \overline{B}_{\mathbb{E}}(0,R)} \|DN(\bar{z})\|_{\mathcal{L}(\mathbb{E},\mathbb{F})} \leq \eta.$$

By the continuity of the solution operator of the linearized problem (see Proposition 1.9), it follows that

$$\begin{aligned}
\|\Phi(z)\|_{\mathbb{E}} &= \|\tilde{\Phi}_{v_0,h_0}(N(z))\|_{\mathbb{E}} \\
&\leq C\big(\|N(z)\|_{\mathbb{F}} + \|v_0\|_{W_p^{2-\frac{2}{p}}(\dot{\mathbb{R}}^n)} + \|h_0\|_{W_p^{3-\frac{2}{p}}(\mathbb{R}^{n-1})}\big) \\
&\leq C\big(\|N(z) - N(0)\|_{\mathbb{F}} + \|v_0\|_{W_p^{2-\frac{2}{p}}(\dot{\mathbb{R}}^n)} + \|h_0\|_{W_p^{3-\frac{2}{p}}(\mathbb{R}^{n-1})}\big) \\
&\leq C\big(\sup_{\bar{z}\in \overline{B}_{\mathbb{E}}(0,R)} \|DN(\bar{z})\|_{\mathcal{L}(\mathbb{E},\mathbb{F})}\|z\|_{\mathbb{E}} + \|v_0\|_{W_p^{2-\frac{2}{p}}(\dot{\mathbb{R}}^n)} + \|h_0\|_{W_p^{3-\frac{2}{p}}(\mathbb{R}^{n-1})}\big) \\
&\leq C\eta R + C\big(\|v_0\|_{W_p^{2-\frac{2}{p}}(\dot{\mathbb{R}}^n)} + \|h_0\|_{W_p^{3-\frac{2}{p}}(\mathbb{R}^{n-1})}\big), \quad z \in \overline{B}_{{}_{v_0,h_0}\mathbb{E}}(0,R),
\end{aligned}$$

as well as

$$\begin{aligned}
\|\Phi(z_2) - \Phi(z_1)\|_{\mathbb{E}} &= \|\tilde{\Phi}_{0,0}(N(z_2) - N(z_1))\|_{\mathbb{E}} \\
&\leq C\|N(z_2) - N(z_1)\|_{\mathbb{F}} \\
&\leq C \sup_{\bar{z}\in \overline{B}_{\mathbb{E}}(0,R)} \|DN(\bar{z})\|_{\mathcal{L}(\mathbb{E},\mathbb{F})}\|z_2 - z_1\|_{\mathbb{E}} \\
&\leq C\eta\|z_2 - z_1\|_{\mathbb{E}}, \quad z_1, z_2 \in \overline{B}_{{}_{v_0,h_0}\mathbb{E}}(0,R).
\end{aligned}$$

Choosing now η (and therefore R) and the initial values sufficiently small, it follows that

$$\Phi(\overline{B}_{{}_{v_0,h_0}\mathbb{E}}(0,R)) \subset \overline{B}_{{}_{v_0,h_0}\mathbb{E}}(0,R),$$

and that Φ is contractive.

It remains to show that $\overline{B}_{{}_{v_0,h_0}\mathbb{E}}(0,R)$ is not empty. We extend $\gamma_\pm v_0$ in time with the extension operator $\mathcal{E}_t$ (see Proposition 1.15). Let v_* be the solution of the problem

$$\left\{\begin{array}{rcll}
\partial_t v_* - \Delta v_* &=& 0 & \text{in } (0,T_0)\times\mathbb{R}^n_\pm, \\
v_* &=& \mathcal{E}_t\gamma v_0 & \text{on } (0,T_0)\times\mathbb{R}^{n-1}, \\
v_*(0) &=& \mathcal{R}_\pm v_0 & \text{in } \mathbb{R}^n_\pm,
\end{array}\right.$$

where $\mathcal{R}_\pm$ is the restriction operator on the half space. We have

$$v_*(0) = v_0, \quad v_* \in \mathbb{E}_u(T_0,\dot{\mathbb{R}}^n) \quad \text{and} \quad \|v_*\|_{\mathbb{E}_u(T,\dot{\mathbb{R}}^n)} \leq C\|v_0\|_{W_p^{2-\frac{2}{p}}(\dot{\mathbb{R}}^n))}.$$

Further, we define $(\tilde{v}_*, \theta_*, h_*) = \tilde{\Phi}_{0,h_0}(0,0,0,0)$. By construction, we deduce that $h_*(0) = h_0$,

$$(\theta_*, h_*) \in \mathbb{E}_\pi(T_0,\dot{\mathbb{R}}^n)\times\mathbb{E}_h(T_0,\mathbb{R}^{n-1}), \quad \text{and} \quad \|\theta_*\|_{\mathbb{E}_\pi(T_0,\dot{\mathbb{R}}^n)} + \|h_*\|_{\mathbb{E}_h(T_0,\mathbb{R}^{n-1})} \leq C\|h_0\|_{W_p^{3-\frac{2}{p}}(\mathbb{R}^{n-1})}.$$

Choosing now the initial values sufficiently small, it follows that $(v_*, \theta_*, h_*) \in \overline{B}_{{}_{v_0,h_0}\mathbb{E}}(0, R)$.

Application of the contraction mapping principle delivers a unique solution of (3.23), provided that v_0 and h_0 are sufficiently small in their corresponding spaces. Since (v, θ, h) is a solution of (3.23), we additionally obtain the regularity of the pressure jump

$$[\![\theta]\!] \in W_p^{\frac{1}{2}-\frac{1}{2p}}(0, T_0; L_p(\mathbb{R}^{n-1})) \cap L_p(0, T_0; W_p^{1-\frac{1}{p}}(\mathbb{R}^{n-1})).$$

It remains to show, that the smallness condition on u_0 of Theorem 3.1 transforms to a smallness condition on v_0. We recall the definition $v_0(x', x_n) = u_0(x', x_n - h_0(x')) = u_0 \circ \Theta_{h_0}^{-1}(x', x_n)$, with $\Theta_{h_0}^{-1}(x', x_n) = (x', x_n - h_0(x'))$. We define the operator $\tilde{\Theta}_0^{-1}$:

$$\tilde{\Theta}_0^{-1} \colon L_p(\mathbb{R}^n) \to L_p(\mathbb{R}^n), \quad \tilde{f} \mapsto \tilde{f} \circ \Theta_{h_0}^{-1}.$$

It holds $\|\tilde{\Theta}_0^{-1} \tilde{f}\|_{p,\mathbb{R}^n} = \|\tilde{f}\|_{p,\mathbb{R}^n}$. By $h_0 \in BUC^2(\mathbb{R}^{n-1})$, it follows that

$$\tilde{\Theta}_0^{-1} \in \mathcal{L}(H_p^2(\mathbb{R}^n \setminus \Gamma_0), H_p^2(\dot{\mathbb{R}}^n))$$

with

$$\|\tilde{\Theta}_0^{-1}\|_{\mathcal{L}(H_p^2(\mathbb{R}^n\setminus\Gamma_0), H_p^2(\dot{\mathbb{R}}^n))} \le C\|h_0\|_{W_\infty^2(\mathbb{R}^{n-1})},$$

and hence

$$\begin{aligned}\|v_0\|_{W_p^{2-\frac{2}{p}}(\dot{\mathbb{R}}^n)} &\le C\big(1 + \|h_0\|_{W_\infty^2(\mathbb{R}^{n-1})}\big)\|u_0\|_{W_p^{2-\frac{2}{p}}(\mathbb{R}^n\setminus\Gamma_0)} \\ &\le C\big(1 + \|h_0\|_{W_p^{3-\frac{2}{p}}(\mathbb{R}^{n-1})}\big)\|u_0\|_{W_p^{2-\frac{2}{p}}(\mathbb{R}^n\setminus\Gamma_0)},\end{aligned}$$

by the embedding $W_p^{3-\frac{2}{p}}(\mathbb{R}^{n-1}) \hookrightarrow BUC^2(\mathbb{R}^{n-1})$. $\square$

Chapter 4

Generalized viscoelastic fluids with a free boundary without surface tension

In this chapter, we will be concerned with a free boundary problem describing the motion of an incompressible generalized viscoelastic fluid without surface tension. Contrary to the previous two chapters, we consider the Lagrangian formulation of the problem. The domain occupied by the fluid is denoted by $\Omega(t) \subset \mathbb{R}^n$ and its outer normal by $\nu(t)$. At the initial configuration, it is assumed that the boundary of the domain $\partial\Omega(0) = \Gamma_F(0) \cup \Gamma_D$ is compact, and decomposes into two disjoint, open, and closed subsets $\Gamma_F(0)$ and Γ_D. The aim is to prove local-in-time solvability for arbitrarily large initial data for the following system:

$$
\left\{
\begin{array}{rcll}
\rho(\partial_t u + u.\nabla u) - \operatorname{Div} 2\alpha(|Eu|^2)Eu + \nabla\pi &=& \operatorname{Div}\mu(\tau) & \text{in } (0,T_0)\times\Omega(t),\\
\operatorname{div} u &=& 0 & \text{in } (0,T_0)\times\Omega(t),\\
\partial_t\tau + u.\nabla\tau &=& g(\nabla u,\tau) & \text{in } (0,T_0)\times\Omega(t),\\
-(2\alpha(|Eu|^2)Eu - \pi)\nu &=& \mu(\tau)\nu & \text{on } (0,T_0)\times\Gamma_F(t),\\
V &=& u\cdot\nu & \text{on } (0,T_0)\times\Gamma_F(t),\\
u &=& 0 & \text{on } (0,T_0)\times\Gamma_D,\\
u(0) &=& u_0 & \text{in } \Omega_0,\\
\tau(0) &=& \tau_0 & \text{in } \Omega_0,\\
\Gamma_F(0) &=& \Gamma_{F,0}. &
\end{array}
\right.
\tag{4.1}
$$

This system will be described as follows: The unknowns are the velocity field u, the pressure π, the elastic part of the stress τ, and the free part of the boundary Γ_F. The symmetric part of the velocity gradient is denoted by $Eu = \frac{1}{2}(\nabla u + (\nabla u)^T)$ and the normal velocity of the free interface Γ_F by V. Given are the constant density of the fluid ρ, the viscosity function $\alpha\colon [0,\infty) \to [0,\infty)$, and two functions $\mu\colon \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ and $g\colon \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$, coupling the elastic part of the stress τ with the velocity field u. The structure conditions

$$\alpha(s) > 0, \quad \alpha(s) + 2s\alpha'(s) > 0, \quad s \geq 0 \quad \text{and} \quad \mu(0) = g(0,0) = 0$$

will play an important role in the investigation of the problem. Two boundary conditions are of interest. We consider a free boundary part Γ_F, which is driven by the motion of the fluid, and a fixed boundary part Γ_D, where we prescribe Dirichlet boundary conditions. Each of this boundary parts can be empty. Furthermore, the initial conditions u_0, τ_0, and $\Gamma_{F,0}$, satisfying the natural compatibility condition

$$\text{(4.2)}\quad \operatorname{div} u_0 = 0 \quad \text{on } \Omega_0, \quad [2\alpha(|Eu_0|^2)Eu_0\nu_0 + \mu(\tau_0)\nu_0]_{\tan} = 0 \quad \text{on } \Gamma_{F,0}, \quad \text{and} \quad u_0 = 0 \quad \text{on } \Gamma_D$$

are given. The initial domain is denoted by $\Omega_0 = \Omega(0)$ and the outer normal on Ω_0 by $\nu_0 = \nu(0)$.

The first equation of (4.1) is the balance of momentum, assuming the stress admits the form

$$\mathcal{S}(u, \pi, \tau) = 2\alpha(|Eu|^2)Eu - \pi + \mu(\tau).$$

Since the density $\rho > 0$ is constant, the second equation characterises the incompressibility of the fluid. The transport equation describes the evolution of the elastic part of the stress. The first boundary condition on the free surface says, that the stress in normal direction $\mathcal{S}(u, \pi, \tau)\nu$ vanishes, and the kinematic condition $V = u \cdot \nu$ expresses the fact, that the free surface Γ_F is transported by the motion of the fluid.

Variants of system (4.1) have been studied intensively in the literature. In the case, that the domain $\Omega(t) = \Omega(0)$ is fixed, i.e. the boundary $\Gamma_{F,0}$ is empty, system (4.1) corresponds to system (2.1), which was investigated in Chapter 2. For an overview on the existing literature, we refer the reader to the aforementioned chapter. The two-phase model in the generalized Newtonian case ($\tau = 0$, α not constant) with surface tension was studied in Chapter 3, using Eulerian coordinates. We refer to this chapter for an overview on the existing literature on the one- and two-phase problems in the Newtonian case in Eulerian coordinates, the two-phase problem in Lagrangian coordinates, and the generalized Newtonian case. The one-phase problem in the Newtonian case ($\tau = 0$, $\alpha > 0$ constant) with and without surface tension $\sigma \geq 0$

$$\text{(4.3)}\qquad \begin{cases} \rho(\partial_t u + u.\nabla u) - \operatorname{Div} 2\alpha Eu + \nabla\pi &= f \qquad \text{in } (0, T_0) \times \Omega(t), \\ \operatorname{div} u &= 0 \qquad \text{in } (0, T_0) \times \Omega(t), \\ -(2\alpha Eu - \pi)\nu &= \sigma\kappa\nu \qquad \text{on } (0, T_0) \times \Gamma_F(t), \\ V &= u \cdot \nu \qquad \text{on } (0, T_0) \times \Gamma_F(t), \\ u &= 0 \qquad \text{on } (0, T_0) \times \Gamma_D, \\ u(0) &= u_0 \qquad \text{in } \Omega_0, \\ \Gamma_F(0) &= \Gamma_{F,0} \end{cases}$$

was considered by various authors, using Lagrangian coordinates. This is a special case of (4.1) if $\sigma = 0$. First, system (4.3) without surface tension ($\sigma = 0$) was investigated by Solonnikov [Sol77b, Sol88] and later by Shibata and Shimizu [SS07a, SS07b]. System (4.3) with surface tension ($\sigma > 0$) was also investigated in a long series of papers by Solonnikov [Sol89, Sol91, Sol99, Sol03a, Sol03b, Sol04], by Tani and Tanaka [TT95], by Tani [Tan96], and recently by Shibata and Shimizu [SS11]. To the author's best knowledge, there is no result on a viscoelastic fluid model with a free surface.

The main result of this chapter is the local-in-time solvability of (4.1) in Lagrangian coordinates for arbitrarily large initial values. To state the main theorem, we need to introduce the formulation of (4.1) in Lagrangian coordinates. We define the transformation

$$\Theta_v \colon (0, T) \times \Omega_0 \to \bigcup_{t \in (0,T)} \{t\} \times \Omega(t), \quad (t, \xi) \mapsto (t, X_v(t, \xi)),$$

with

$$X_v(t, \xi) = \xi + \int_0^t v(s, \xi) ds, \quad (t, \xi) \in (0, T) \times \Omega_0,$$

where $v(t,\xi) = u(t, X(t,\xi))$ Passing to Lagrangian coordinates in (4.1) and setting $(v,\theta,\eta) = (u,\pi,\tau)\circ\Theta_v$ (the transformation to Lagrangian coordinates is discussed in the next section (Section 4.1)), we obtain a problem on the fixed initial domain Ω_0

$$
\left\{
\begin{array}{rcll}
\rho\partial_t v - \operatorname{Div} 2\alpha(|Ev|^2)Ev + \nabla\theta &=& \bar{F}(v,\theta,\eta) & \text{in } (0,T_0)\times\Omega_0,\\
\operatorname{div} v &=& \bar{F}_d(v) & \text{in } (0,T_0)\times\Omega_0,\\
\partial_t\eta &=& \bar{G}(v,\eta) & \text{in } (0,T_0)\times\Omega_0,\\
-(2\alpha(|Ev|^2)Ev-\theta)\nu_0 &=& \bar{H}(v,\theta,\eta) & \text{on } (0,T_0)\times\Gamma_{F,0},\\
v &=& 0 & \text{on } (0,T_0)\times\Gamma_D,\\
v(0) &=& u_0 & \text{in } \Omega_0,\\
\eta(0) &=& \tau_0 & \text{in } \Omega_0,
\end{array}
\right.
\tag{4.4}
$$

where $\bar{F}$, $\bar{F}_d$, $\bar{G}$, and $\bar{H}$ are nonlinearities occurring through the transformation in Lagrangian coordinates. These nonlinearities are calculated in the next section, see (4.7), (4.16), (4.18), (4.20), and (4.25) for their definitions. We discuss local-in-time solvability of (4.4) instead of (4.1). We prove the following theorem:

Theorem 4.1. *Fix $n\in\mathbb{N}$, $n\geq 2$, $n+2<p<\infty$, and $T_0,\rho>0$. Let Ω_0 be a domain with a compact $C^{2,1}$-boundary, such that boundary $\partial\Omega_0 = \Gamma_{F,0}\cup\Gamma_D$ decomposes in two disjoint subsets $\Gamma_{F,0}$ and Γ_D, which are open and closed in $\partial\Omega_0$. Assume that $\alpha\in C^3([0,\infty))$, $\mu\in C^3(\mathbb{R}^{n\times n},\mathbb{R}^{n\times n})$, and $g\in C^2(\mathbb{R}^{n\times n}\times\mathbb{R}^{n\times n},\mathbb{R}^{n\times n})$ satisfy the structure conditions*

$$\alpha(s)>0,\quad \alpha(s)+2s\alpha'(s)>0,\quad s\geq 0 \quad \textit{and} \quad \mu(0)=g(0,0)=0.$$

Then, for each $u_0\in W_p^{2-\frac{2}{p}}(\Omega_0)$ and $\tau_0\in H_p^1(\Omega_0)$, satisfying the compatibility conditions

$$\operatorname{div} u_0 = 0 \quad \textit{in } \Omega_0,\quad [2\alpha(|Eu_0|^2)Eu_0\nu_0+\mu(\tau_0)\nu_0]_{tan}=0 \quad \textit{on } \Gamma_{F,0},\quad \textit{and}\quad u_0=0 \quad \textit{on } \Gamma_D,$$

there exists a time $T\in(0,T_0)$ and a unique strong solution (v,θ,η) of (4.4) on the time interval $(0,T)$ in the regularity class

$$
\begin{array}{lcl}
v & \in & H_p^1(0,T;L_p(\Omega_0))\cap L_p(0,T;H_p^2(\Omega_0)),\\
\theta & \in & L_p(0,T;\widehat{H}_p^1(\Omega_0)),\\
\gamma_{\Gamma_{F_0}}\theta & \in & W_p^{\frac{1}{2}-\frac{1}{2p}}(0,T;L_p(\Gamma_{F,0}))\cap L_p(0,T;W_p^{1-\frac{1}{p}}(\Gamma_{F,0})),\\
\eta & \in & W_\infty^1(0,T;L_p(\Omega_0))\cap H_p^1(0,T;H_p^1(\Omega_0)).
\end{array}
$$

Sketch of the proof

Let us present the main ideas of the proof. System (4.1) is solved in Lagrangian coordinates. If a velocity field v in Lagrangian coordinates is given, the connection between Eulerian coordinates $x\in\Omega(t)$ and Lagrangian coordinates $\xi\in\Omega_0$ at a time $t\in(0,T)$ is determined by the formula

$$x = X_v(t,\xi) = \xi + \int_0^t v(s,\xi)ds$$

and the Eulerian velocity field is determined by $v(t,\xi) = u(t,X_v(t,\xi))$. We define the transformation

$$\Theta_v\colon (0,T)\times\Omega_0 \to \bigcup_{t\in(0,T)}\{t\}\times\Omega(t),\quad (t,\xi)\mapsto(t,X_v(t,\xi)),$$

and the unknowns $(v,\theta,\eta) = (u,\pi,\tau)\circ\Theta_v$ in Lagrangian coordinates. The transformed system can be written in the form of (4.4) with nonlinearities $\bar{F}, \bar{F}_d, \bar{G}$, and $\bar{H}$. There are two main advantages of Lagrangian approach. Firstly, the transport term $u.\nabla\tau$ in the transport equation vanishes due to $(\partial_t\tau + u.\nabla\tau)\circ\Theta_v = \partial_t\eta$ and therefore, we are in a position to apply the standard version of the contraction mapping principle (see page 126). Secondly, the problem is reduced to a problem on the fixed initial domain.

System (4.4) is a quasilinear problem on the fixed initial domain Ω_0. Since we are interested in solutions for arbitrarily large initial data, it is convenient to reduce (4.4) to $u_0 = 0$ and $\tau_0 = 0$. In this situation, the embedding constants in the proposition on embedding theorems (Proposition 1.14) are independent of T, $0 < T < T_0$. For this purpose, we define functions (v_*, θ_*, η_*) with

$$\begin{array}{lcl} v_* & \in & H_p^1(0,T_0;L_p(\Omega_0)) \cap L_p(0,T_0;H_p^2(\Omega_0)), \\ \theta_* & \in & L_p(0,T_0;\widehat{H}_p^1(\Omega_0)), \\ \gamma_{\Gamma_{F_0}}\theta_* & \in & W_p^{\frac{1}{2}-\frac{1}{2p}}(0,T_0;L_p(\Gamma_{F,0})) \cap L_p(0,T_0;W_p^{1-\frac{1}{p}}(\Gamma_{F,0})), \\ \eta_* & \in & L_\infty(0,T_0;H_p^1(\Omega_0)) \cap W_\infty^1(0,T_0;L_p(\Omega_0)), \end{array}$$

and $(v_*(0), \eta_*(0)) = (u_0, \tau_0)$. In the subsection on the generalized Stokes equation (Subsection 1.2.3), we introduced the quasilinear operator $\mathcal{A}(Ev_*)$, with $\mathcal{A}(Ev)v = -2\operatorname{Div}\alpha(|Ev|^2)Ev$, as well as the corresponding Neumann boundary operator $\mathcal{B}_{\mathcal{N}}(Ev_*)$. Setting

$$(w + v_*, \psi + \theta_*, \zeta + \tau_0) = (v, \theta, \eta),$$

we write (4.4) in the equivalent form

$$\tag{4.5} \left\{ \begin{array}{rcll} \rho\partial_t w + \mathcal{A}(Ev_*)w + \nabla\psi & = & f_* + F(w,\psi,\zeta) & \text{in } (0,T_0)\times\Omega_0, \\ \operatorname{div} w & = & F_d(w) & \text{in } (0,T_0)\times\Omega_0, \\ \partial_t\zeta & = & G(w,\zeta) & \text{in } (0,T_0)\times\Omega_0, \\ \mathcal{B}_{\mathcal{N}}(Ev_*)(w,\psi) & = & h_* + H(w,\psi,\zeta) & \text{on } (0,T_0)\times\Gamma_{F,0}, \\ w & = & 0 & \text{on } (0,T_0)\times\Gamma_D, \\ w(0) & = & 0 & \text{in } \Omega_0, \\ \zeta(0) & = & 0 & \text{in } \Omega_0, \end{array} \right.$$

where we additionally reduced the initial values to zero. The operator $\mathcal{A}(Ev_*)$ is fixed and has time a spatial dependent coefficient, the operator $\mathcal{B}_{\mathcal{N}}(Ev_*)$ is the corresponding Neumann boundary operator, f_*, h_* are given functions, and F, F_d, G, and H are given nonlinearities. The associated linearization of (4.5) is

$$\tag{4.6} \left\{ \begin{array}{rcll} \rho\partial_t w + \mathcal{A}(Ev_*)w + \nabla\psi & = & \bar{f} & \text{in } (0,T_0)\times\Omega_0, \\ \operatorname{div} w & = & \bar{f}_d & \text{in } (0,T_0)\times\Omega_0, \\ \partial_t\zeta & = & \bar{g} & \text{in } (0,T_0)\times\Omega_0, \\ \mathcal{B}_{\mathcal{N}}(Ev_*)(w,\psi) & = & \bar{h} & \text{on } (0,T_0)\times\Gamma_{F,0}, \\ w & = & 0 & \text{on } (0,T_0)\times\Gamma_D, \\ w(0) & = & 0 & \text{in } \Omega_0, \\ \zeta(0) & = & 0 & \text{in } \Omega_0, \end{array} \right.$$

where the right-hand sides are given.

We emphasize that the elastic part of the stress in (4.6) is given by $\zeta(t) = \int_0^t \bar{g}(s)ds$. The velocity field and the pressure are given as the solution of a generalized Stokes equation with Neumann boundary condition on $\Gamma_{F,0}$ and Dirichlet boundary conditions on Γ_D. The nonlinear problem (4.5) is solved using the contraction mapping principle.

4.1 Formulation in Lagrangian coordinates

We investigate (4.1) using Lagrangian coordinates. Unless stated otherwise, it is

$$t \in (0,T), \quad x \in \Omega(t), \quad \text{and} \quad \xi \in \Omega_0.$$

We recall the definition of the transformation

$$\Theta_v \colon (0,T) \times \Omega_0 \to \bigcup_{t\in(0,T)} \{t\} \times \Omega(t), \quad (t,\xi) \mapsto (t, X_v(t,\xi)) \quad \text{with} \quad X_v(t,\xi) = \xi + \int_0^t v(s,\xi)ds,$$

where $v(t,\xi) := u(t, X_v(t,\xi))$. Moreover, we define the transformed pressure $\theta(t,\xi) := \pi(t, X_v(t,\xi))$ and the transformed elastic part of the stress $\eta(t,\xi) := \tau(t, X_v(t,\xi))$. We denote the inverse transformation of $X_v(t,\,\cdot\,)$ by $\Xi_v(t,\,\cdot\,)$, i.e.

$$X_v(t, \Xi_v(t,x)) = x \quad \text{and} \quad \Xi_v(t, X_v(t,\xi)) = \xi.$$

The Jacobi matrix of X_v and Ξ_v in the spatial components is denoted by $J_{X_v}(t,\xi)$ and $J_{\Xi_v}(t,x)$ respectively, i.e.

$$\big(J_{X_v}(t,\xi)\big)_{j,k} = \partial_k X_{u,j}(t,\xi) = \delta_{j,k} + \int_0^t \partial_k v_j(\xi,s)ds \quad \text{and} \quad \big(J_{\Xi_v}(t,x)\big)_{j,k} = \partial_k \Xi_{v,j}(t,x), \qquad j,k = 1,\dots,n.$$

Shibata and Shimizu [SS07b, (A.5)] proved the existence of a smooth function $B\colon \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$, with $B(0) = 0$, such that

$$A(t,\xi) := J_{X_v}(t,\xi)^{-1} = J_{\Xi_v}(t, X_v(t,\xi)) = 1 + B(I(v)(t,\xi)), \tag{4.7}$$

where

$$I(v)(t,\xi) := \int_0^t \nabla v(s,\xi)ds. \tag{4.8}$$

To transform the balance of momentum (the first equation of (4.1)) onto a fixed domain, we apply the transformation Θ_v

$$\begin{aligned}\rho(\partial_t u + u.\nabla u)\circ\Theta_v - (\operatorname{Div} 2\alpha(|Eu|^2)Eu)\circ\Theta_v + (\nabla\pi)\circ\Theta_v& \\ = (\operatorname{Div}\mu(\tau))\circ\Theta_v \quad \text{in } (0,T_0)\times\Omega_0.&\end{aligned} \tag{4.9}$$

We analyse each summand separately. It holds

$$
\begin{aligned}
\partial_t v(t,\xi) = \partial_t(u(t,X_v(t,\xi))) &= (\partial_t u)(t,X_v(t,\xi)) + \sum_{j=1}^n (\partial_j u)(t,X_v(t,\xi))\partial_t X_{u,j}(t,\xi) \\
&= (\partial_t u)(t,X_v(t,\xi)) + \sum_{j=1}^n (\partial_j u)(t,X_v(t,\xi)) u_j(t,X_v(t,\xi)) \\
&= (\partial_t u)(t,X_v(t,\xi)) + (u.\nabla u)(t,X_v(t,\xi)),
\end{aligned}
$$

and hence

$$
\partial_t v = (\partial_t u)\circ\Theta_v + (u\circ\Theta_v).(\nabla u)\circ\Theta_v = (\partial_t u + u.\nabla u)\circ\Theta_v. \tag{4.10}
$$

Next, we investigate $(\operatorname{Div} 2\alpha(|Eu|^2)Eu)\circ\Theta_v$. For this purpose, we compute on account of (4.7)

$$
\begin{aligned}
\partial_j(v(t,\Xi_v(t,x))) &= \sum_{l=1}^n (\partial_l v)(t,\Xi_v(t,x))\partial_j \Xi_{v,l}(t,x) = \sum_{l=1}^n (\partial_l v)(t,\Xi_v(t,x)) J_{\Xi_v}(t,x)_{l,j} \\
&= \sum_{l=1}^n (\partial_l v)(t,\Xi_v(t,x)) A_{l,j}(t,\Xi_v(t,x)) \\
&= (\partial_j v)(t,\Xi_v(t,x)) + \sum_{l=1}^n (\partial_l v)(t,\Xi_v(t,x)) B_{l,j}(I(v)(t,\Xi_v(t,x))).
\end{aligned}
$$

By the previous equation, we conclude that

$$
\begin{aligned}
(\nabla u)\circ\Theta_v &= \big((\partial_j u_k)\circ\Theta_v\big)_{j,k=1,\dots,n} = \big(\partial_j v_k + \sum_{l=1}^n B(I(v))_{l,j}\partial_l v_k\big)_{j,k=1,\dots,n} \\
&= \nabla v + B(I(v))^T\nabla v,
\end{aligned} \tag{4.11}
$$

and hence, the transformed symmetric part of the gradient reads

$$
\mathcal{E}(v) := (Eu)\circ\Theta_v = Ev + \frac{1}{2}\big(B(I(v))^T\nabla v + (\nabla v)^T B(I(v))\big). \tag{4.12}
$$

In the same way we transformed the gradient, we proceed with the divergence. But we consider a general situation, where F is a matrix-valued function and $G := F\circ\Theta_v$. Then, taking into account $\operatorname{Div}(B(I(v))^T) = 0$ (see [SS07b, A.6]) and (4.11), it follows that

$$
\begin{aligned}
(\operatorname{Div} F)\circ\Theta_v &= \big(\sum_{j=1}^n (\partial_j F_{k,j})\circ\Theta_v\big)_{k=1,\dots,n} = \big(\sum_{j=1}^n \partial_j G_{k,j} + \sum_{j,l=1}^n B(I(v))_{l,j}\partial_l G_{k,j}\big)_{k=1,\dots,n} \\
&= \big(\sum_{j=1}^n \partial_j G_{k,j} + \sum_{j,l=1}^n \partial_l(B(I(v))_{l,j}G_{k,j})\big)_{k=1,\dots,n} = \operatorname{Div} G + \operatorname{Div}(GB(I(v))^T).
\end{aligned} \tag{4.13}
$$

Applying this equality to $F = \operatorname{Div} 2\alpha(|Eu|^2)Eu$ and using the definition of $\mathcal{E}(v)$ (see (4.12)), we infer

$$
\begin{aligned}
&\big(\operatorname{Div} 2\alpha(|Eu|^2)Eu\big)\circ\Theta_v\\
&\quad= \operatorname{Div} 2\alpha(|\mathcal{E}(v)|^2)\mathcal{E}(v) + \operatorname{Div}\big(2\alpha(|\mathcal{E}(v)|^2)\mathcal{E}(v)B(I(v))^T\big)\\
&\quad= \operatorname{Div} 2\alpha(|\mathcal{E}(v)|^2)Ev + \operatorname{Div}\Big(\alpha(|\mathcal{E}(v)|^2)\big(B(I(v))^T\nabla v + (\nabla v)^T B(I(v)) + \nabla v B(I(v))^T\\
&\qquad + (\nabla v)^T B(I(v))^T + B(I(v))^T\nabla v B(I(v))^T + (\nabla v)^T B(I(v))B(I(v))^T\big)\Big)\\
&\quad= \operatorname{Div} 2\alpha(|\mathcal{E}(v)|^2)Ev + \bar{F}^{(1)}(v),
\end{aligned}
$$

with

$$
\begin{aligned}
\bar{F}^{(1)}(v) := \operatorname{Div}\Big(&\alpha(|\mathcal{E}(v)|^2)\big(B(I(v))^T\nabla v + (\nabla v)^T B(I(v)) + \nabla v B(I(v))^T + (\nabla v)^T B(I(v))^T\\
&+ B(I(v))^T\nabla v B(I(v))^T + (\nabla v)^T B(I(v))B(I(v))^T\big)\Big).
\end{aligned} \tag{4.14}
$$

Next, we investigate $(\nabla\pi)\circ\Theta_v$ as well as $(\operatorname{Div}\mu(\tau))\circ\Theta_v$. By (4.11), we have

$$
(\nabla\pi)\circ\Theta_v = \nabla\theta + B(I(v))^T\nabla\theta,
$$

and, from the first line of (4.13), we deduce that

$$
(\operatorname{Div}\mu(\tau))\circ\Theta_v = \operatorname{Div}\mu(\eta) + \big(\sum_{j,l=1}^{n} B(I(v))_{l,j}\partial_l\mu(\eta)_{k,j}\big)_{k=1,\dots,n}.
$$

Summarized, (4.9) reduces to

$$
\begin{aligned}
&\rho\partial_t v - \operatorname{Div} 2\alpha(|\mathcal{E}(v)|^2)Ev - \bar{F}^{(1)}(v) + \nabla\theta + B(I(v))^T\nabla\theta\\
&\qquad\qquad = \operatorname{Div}\mu(\eta) + \big(\sum_{j,l=1}^{n} B(I(v))_{l,j}\partial_l\mu(\eta)_{k,j}\big)_{k=1,\dots,n} \quad \text{in } (0,T_0)\times\Omega_0.
\end{aligned}
$$

We write this equation in the form

$$
\rho\partial_t v - \operatorname{Div} 2\alpha(|Ev|^2)Ev + \nabla\theta = \bar{F}(v,\theta,\eta) \quad \text{in } (0,T_0)\times\Omega_0, \tag{4.15}
$$

with

$$
\begin{aligned}
\bar{F}(v,\theta,\eta) &:= \operatorname{Div} 2(\alpha(|\mathcal{E}(v)|^2) - \alpha(|Ev|^2))Ev + \bar{F}^{(1)}(v) - B(I(v))^T\nabla\theta\\
&\qquad + \operatorname{Div}\mu(\eta) + \big(\sum_{j,l=1}^{n} B(I(v))_{l,j}\partial_l\mu(\eta)_{k,j}\big)_{k=1,\dots,n},
\end{aligned} \tag{4.16}
$$

where $\bar{F}^{(1)}(v)$ is defined in (4.14).

According to the first and second line of (4.13), the divergence free condition transforms to

$$
\operatorname{div} v = \bar{F}_d(v) \quad \text{in } (0,T_0)\times\Omega_0, \tag{4.17}
$$

with

$$
\bar{F}_d(v) = -\operatorname{div}(B(I(v))v) = -(\nabla v : B(I(v))). \tag{4.18}
$$

Our next subject is the transformation of the transport equation (the third line of (4.1)). Similar to the argumentation (4.10), it holds

$$\partial_t \eta = (\partial_t \tau) \circ \Theta_v + (u \circ \Theta_v).(\nabla \tau) \circ \Theta_v = (\partial_t \tau + u.\nabla \tau) \circ \Theta_v. \tag{4.19}$$

Combining the previous equation with (4.11), the transport equation in Lagrangian coordinates takes the form

$$\partial_t \eta = \bar{G}(v, \eta) \quad \text{in } (0, T_0) \times \Omega_0,$$

with

$$\bar{G}(v, \eta) = g(\nabla v + B(I(v))^T \nabla v, \eta). \tag{4.20}$$

Next, we transform the first boundary condition on the free surface $\Gamma_F(t)$. For $\mathbb{R}^n$-valued functions f and $g := f \circ \Theta_v$, Shibata and Shimizu [SS07b, (A.11)] proved

$$f \cdot \nu = 0 \quad \text{on } (0, T_0) \times \Gamma_F(t) \quad \text{if and only if} \quad g^T A^T \nu_0 = 0 \quad \text{on } (0, T_0) \times \Gamma_{F,0}.$$

This implies, for matrix-valued functions F and $G := F \circ \Theta_v$,

$$F\nu = (\sum_{j=1}^n F_{l,j} \nu_j)_{l=1,\dots,n} = 0 \quad \text{on } (0, T_0) \times \Gamma_F(t)$$

if and only if

$$(\sum_{j,k=1}^n G_{l,j} A_{k,j} \nu_{0,k})_{l=1,\dots,n} = G A^T \nu_0 = 0 \quad \text{on } (0, T_0) \times \Gamma_{F,0}.$$

Hence, by $A = 1 + B(I(v))$, the boundary condition (see the fourth line of (4.1))

$$-(2\alpha(|Eu|^2) Eu - \pi)\nu = \mu(\tau)\nu \quad \text{on } (0, T_0) \times \Gamma_F(t),$$

is in Lagrangian coordinates given by

$$-\big(2\alpha(|\mathcal{E}(v)|^2)\mathcal{E}(v) - \theta\big)\big(1 + B(I(v))^T\big)\nu_0 = \mu(\eta)\big(1 + B(I(v))^T\big)\nu_0 \quad \text{on } (0, T_0) \times \Gamma_{F,0}. \tag{4.21}$$

To shorten notation, we define

$$\begin{aligned} \bar{H}^{(1)}(v) := -\alpha(|\mathcal{E}(v)|^2)\big(&B(I(v))^T \nabla v + (\nabla v)^T B(I(v)) + \nabla v B(I(v))^T + (\nabla v)^T B(I(v))^T \\ &+ B(I(v))^T \nabla v B(I(v))^T + (\nabla v)^T B(I(v)) B(I(v))^T\big)\nu_0. \end{aligned} \tag{4.22}$$

Using this abbreviation and $2\mathcal{E}(v) = 2Ev + B(I(v))^T \nabla v + (\nabla v)^T B(I(v))$, (4.21) reads

$$-2\alpha(|\mathcal{E}(v)|^2) Ev\nu_0 + \bar{H}^{(1)}(v) + \theta\big(1 + B(I(v))^T\big)\nu_0 = \mu(\eta)\big(1 + B(I(v))^T\big)\nu_0 \quad \text{on } (0, T_0) \times \Gamma_{F,0}. \tag{4.23}$$

The transformed boundary condition (4.23) is given by

$$-(2\alpha(|Ev|^2) Ev - \theta)\nu_0 = \bar{H}(v, \theta, \eta) \quad \text{on } (0, T_0) \times \Gamma_{F,0}, \tag{4.24}$$

with

$$\bar{H}(v,\theta,\eta) := 2(\alpha(|\mathcal{E}(v)|^2) - \alpha(|Ev|^2))Ev\nu_0 - \bar{H}^{(1)}(v) - B(I(v))^T\theta\nu_0 \\ + \mu(\eta)\big(1 + B(I(v))^T\big)\nu_0, \tag{4.25}$$

where $\bar{H}^{(1)}(v)$ is defined in (4.22).

In summary, the transformed system (4.4) reads

$$\begin{cases} \rho\partial_t v - \operatorname{Div} 2\alpha(|Ev|^2)Ev + \nabla\theta &= \bar{F}(v,\theta,\eta) & \text{in } (0,T_0)\times\Omega_0,\\ \operatorname{div} v &= \bar{F}_d(v) & \text{in } (0,T_0)\times\Omega_0,\\ \partial_t\eta &= \bar{G}(v,\eta) & \text{in } (0,T_0)\times\Omega_0,\\ -(2\alpha(|Ev|^2)Ev - \theta)\nu_0 &= \bar{H}(v,\theta,\eta) & \text{on } (0,T_0)\times\Gamma_{F,0},\\ v &= 0 & \text{on } (0,T_0)\times\Gamma_D,\\ v(0) &= u_0 & \text{in } \Omega_0,\\ \eta(0) &= \tau_0 & \text{in } \Omega_0. \end{cases}$$

We recall the definition of the nonlinearities $\bar{F}$, $\bar{F}_d$, $\bar{G}$, and $\bar{H}$. The term $\bar{F}$ is defined by (see (4.7) and (4.16))

$$\bar{F}(v,\theta,\eta) = \operatorname{Div} 2(\alpha(|\mathcal{E}(v)|^2) - \alpha(|Ev|^2))Ev + \bar{F}^{(1)}(v) - B(I(v))^T\nabla\theta \\ + \operatorname{Div}\mu(\eta) + \Big(\sum_{j,l=1}^{n} B(I(v))_{l,j}\partial_l\mu(\eta)_{k,j}\Big)_{k=1,\dots,n},$$

with (see (4.14))

$$\bar{F}^{(1)}(v) = \operatorname{Div}\Big(\alpha(|\mathcal{E}(v)|^2)\big(B(I(v))^T\nabla v + (\nabla v)^T B(I(v)) + \nabla v B(I(v))^T + (\nabla v)^T B(I(v))^T \\ + B(I(v))^T\nabla v B(I(v))^T + (\nabla v)^T B(I(v))B(I(v))^T\big)\Big).$$

The nonlinearity $\bar{F}_d$ is given by (see (4.18))

$$\bar{F}_d(v) = -\operatorname{div}(B(I(v))v) = -(\nabla v : B(I(v))),$$

and $\bar{G}$ by (see (4.20))

$$\bar{G}(v,\eta) = g(\nabla v + B(I(v))^T\nabla v, \eta).$$

Further, the right-hand side on the boundary $\bar{H}$ is (see (4.25))

$$\bar{H}(v,\theta,\eta) = 2(\alpha(|\mathcal{E}(v)|^2) - \alpha(|Ev|^2))Ev\nu_0 - \bar{H}^{(1)}(v) - B(I(v))^T\theta\nu_0 \\ + \mu(\eta)\big(1 + B(I(v))^T\big)\nu_0,$$

with (see (4.22))

$$\bar{H}^{(1)}(v) = -\alpha(|\mathcal{E}(v)|^2)\big(B(I(v))^T\nabla v + (\nabla v)^T B(I(v)) + \nabla v B(I(v))^T + (\nabla v)^T B(I(v))^T \\ + B(I(v))^T\nabla v B(I(v))^T + (\nabla v)^T B(I(v))B(I(v))^T\big)\nu_0.$$

4.2 Proof of the main theorem

Proof of Theorem 4.1. We give a proof of Theorem 4.1. This theorem states the local-in-time solvability of (4.4).

Reduction to $u_0 = 0$ and $\tau_0 = 0$

In a first step, we reduce (4.4) to $u_0 = 0$ and $\tau_0 = 0$. Similar to the proof of Theorem 2.1, we define function v_* with $v_*(0) = u_0$ as the solution of a Stokes problem. In order to satisfy the compatibility conditions in this Stokes problem (see Proposition 1.8), we define

$$h = \mathcal{E}_t 2(\alpha(0) - \alpha(|Eu_0|^2))Eu_0\nu_0 \in W_p^{\frac{1}{2}-\frac{1}{2p}}(0,T_0;L_p(\partial\Omega_0)) \cap L_p(0,T_0;W_p^{1-\frac{1}{p}}(\partial\Omega_0)), \tag{4.26}$$

where $\mathcal{E}_t\colon W_p^{1-\frac{3}{p}}(\partial\Omega_0) \to W_p^{\frac{1}{2}-\frac{1}{2p}}(0,T_0;L_p(\partial\Omega_0)) \cap L_p(0,T_0;W_p^{1-\frac{1}{p}}(\partial\Omega_0))$ is the extension operator introduced in the proposition on trace and extension operators (Proposition 1.15). It should be noted, that $(\alpha(0) - \alpha(|Eu_0|^2)) \in W_p^{1-\frac{3}{p}}(\partial\Omega_0)$, due to the proposition on Nemytskij operators (Proposition 1.17). Then, thanks to the compatibility conditions on the boundary $\Gamma_{F,0}$, i.e.

$$-2\alpha(|Eu_0|^2)[Eu_0\nu_0]_{\tan} = [\mu(\tau_0)\nu_0]_{\tan} \quad \text{on } \Gamma_{F,0},$$

we have

$$2\alpha(0)[Eu_0\nu_0]_{\tan} = [h(0) - \mu(\tau_0)\nu_0]_{\tan} \quad \text{on } \Gamma_{F,0},$$

which guarantees the existence of a unique solution

$$\begin{aligned}(v_*,\theta_*) \in \big\{(v,\theta) \in H_p^1(0,T_0;L_p(\Omega_0)) \cap L_p(0,T_0;H_p^2(\Omega_0)) \times L_p(0,T_0;\widehat{H}_p^1(\Omega_0))\colon \\ \gamma_{\Gamma_N}\theta \in W_p^{\frac{1}{2}-\frac{1}{2p}}(0,T_0;L_p(\Gamma_{F,0})) \cap L_p(0,T_0;W_p^{1-\frac{1}{p}}(\Gamma_{F,0}))\big\}\end{aligned} \tag{4.27}$$

to

$$\left\{\begin{array}{rcll} \rho\partial_t v_* - \alpha(0)\Delta v_* + \nabla\theta_* &=& 0 & \text{in } (0,T_0)\times\Omega_0, \\ \operatorname{div} v_* &=& 0 & \text{in } (0,T_0)\times\Omega_0, \\ (2\alpha(0)Ev_* - \theta_*)\nu_0 &=& h - \mu(\tau_0)\nu_0 & \text{on } (0,T_0)\times\Gamma_{F,0}, \\ v_* &=& 0 & \text{on } (0,T_0)\times\Gamma_D, \\ v_*(0) &=& u_0 & \text{in } \Omega_0, \end{array}\right. \tag{4.28}$$

by the proposition on the generalized Stokes operator (Proposition 1.8). We set

$$v = w + v_*, \quad \theta = \psi + \theta_*, \quad \text{and} \quad \eta = \zeta + \tau_0.$$

We use the notation introduced in the subsection on the generalized Stokes operator (Subsection 1.2.3). We recall the definition of the quasilinear operator $\mathcal{A}(Ev_*)w$, i.e.

$$\mathcal{A}(Ev_*)w = -\Big(\sum_{k,l,m=1}^{n} \mathcal{A}_{j,k}^{l,m}(Ev_*)\partial_l\partial_m w_k\Big)_{j=1,\dots,n},$$

with

$$\mathcal{A}_{j,k}^{l,m}(Ev_*) = \alpha(|Ev_*|^2)(\delta_{l,m}\delta_{j,k} + \delta_{j,m}\delta_{k,l}) + 4\alpha'(|Ev_*|^2)(Ev_*)_{j,l}(Ev_*)_{k,m}, \quad j,k,l,m = 1,\dots,n,$$

and the definition of the Neumann boundary operator

$$\mathcal{B}_{\mathcal{N}}(Ev_*)(w,\psi) = \big(\sum_{k,l,m=1}^{n} \mathcal{A}_{j,k}^{l,m}(Ev_*)\nu_{0,l}\partial_m w_k\big)_{j=1,\dots,n} - \psi\nu_0.$$

Our aim is now to formulate an equation for (w,ψ,ζ), which is equivalent to (4.4).

Taking into account $(v,\theta,\eta) = (w+v_*, \psi+\theta_*, \zeta+\tau_0)$ and (see (1.4))

$$-\operatorname{Div}\alpha(|E(w+v_*)|^2)E(w+v_*) = \mathcal{A}(E(w+v_*))(w+v_*),$$

we rewrite the first line of (4.4), i.e.

$$\rho\partial_t v - \operatorname{Div} 2\alpha(|Ev|^2)Ev + \nabla\theta = \bar{F}(v,\theta,\eta) \quad \text{in } (0,T_0)\times\Omega_0$$

in the form

$$\rho\partial_t w + \mathcal{A}(Ev_*)w + \nabla\psi = f_* + F(w,\psi,\zeta) \quad \text{in } (0,T_0)\times\Omega_0,$$

with

$$f_* := -\rho\partial_t v_* - \nabla\theta_* - \mathcal{A}(Ev_*)v_* + \operatorname{Div}\mu(\tau_0),$$

and

$$F(w,\psi,\zeta) := (\mathcal{A}(Ev_*) - \mathcal{A}(E(w+v_*)))(w+v_*) + \bar{F}(w+v_*, \psi+\theta_*, \zeta+\tau_0) - \operatorname{Div}\mu(\tau_0).$$

By (see (4.28))

$$\rho\partial_t v_* - \alpha(0)\Delta u_* + \nabla\theta_* = 0,$$

we have

$$f_* = -\alpha(0)\Delta v_* - \mathcal{A}(Ev_*)v_* + \operatorname{Div}\mu(\tau_0). \tag{4.29}$$

This implies $f_* \in L_p(0,T_0;L_p(\Omega_0))$. Moreover, the representation of F is not suitable for our purposes. According to the definition of $\bar{F}$ (see (4.16)), we decompose F in three parts

$$F(w,\psi,\zeta) = F_w(w) + F_\psi(w,\psi) + F_\zeta(w,\zeta) \tag{4.30}$$

with

$$F_w(w) := \mathcal{A}(Ev_*)(w+v_*) + \operatorname{Div} 2\alpha(|\mathcal{E}(w+v_*)|^2)E(w+v_*) + \bar{F}^{(1)}(w+v_*),$$

$$F_\psi(w,\psi) := -B(I(w+v_*))^T\nabla(\psi+\theta_*), \tag{4.31}$$

$$F_\zeta(w,\zeta) := \operatorname{Div}(\mu(\zeta+\tau_0) - \mu(\tau_0)) + \big(\sum_{j,l=1}^{n} B(I(w+v_*))_{l,j}\partial_l\mu(\zeta+\tau_0)_{k,j}\big)_{k=1,\dots,n}. \tag{4.32}$$

In the definition of F_w, the terms $\mathcal{A}(Ev_*)(w+v_*)$ and $\operatorname{Div} 2\alpha(|\mathcal{E}(w+v_*)|^2)E(w+v_*)$ will not be small separately, but only the sum of both. Inserting the definition of $\mathcal{A}(Ev_*)(w+v_*)$, writing for short notation $v = w + v_*$, and using $2Ev = (\partial_k v_l + \partial_l v_k)_{k,l=1,\ldots,n}$, we have

$$
\begin{aligned}
&(\mathcal{A}(Ev_*)v)_j + (\operatorname{Div} 2\alpha(|\mathcal{E}(v)|^2)E(v))_j \\
&\quad = \sum_{k,l,m=1}^{n} -\mathcal{A}_{j,k}^{l,m}(Ev_*)\partial_l\partial_m v_k + \sum_{l=1}^{n} \partial_l(2\alpha(|\mathcal{E}(v)|^2)(Ev)_{j,l}) \\
&\quad = \sum_{k,l,m=1}^{n} -\mathcal{A}_{j,k}^{l,m}(Ev_*)\partial_l\partial_m v_k + \sum_{l=1}^{n} \alpha(|\mathcal{E}(v)|^2)(\partial_l^2 v_j + \partial_l\partial_j v_l) + 4\alpha'(|\mathcal{E}(v)|^2)(\mathcal{E}(v) \colon \partial_l \mathcal{E}(v))(Ev)_{j,l} \\
&\quad = \sum_{k,l,m=1}^{n} -\mathcal{A}_{j,k}^{l,m}(Ev_*)\partial_l\partial_m v_k + \sum_{l=1}^{n} \alpha(|\mathcal{E}(v)|^2)(\partial_l^2 v_j + \partial_l\partial_j v_l) + 4\alpha'(|\mathcal{E}(v)|^2)(\mathcal{E}(v) \colon \partial_l Ev)\mathcal{E}(v)_{j,l} \\
&\qquad + \sum_{l=1}^{n} 4\alpha'(|\mathcal{E}(v)|^2)(\mathcal{E}(v) \colon \partial_l(\mathcal{E}(v) - Ev))(Ev)_{j,l} - 4\alpha'(|\mathcal{E}(v)|^2)(\mathcal{E}(v) \colon \partial_l Ev)(\mathcal{E}(v) - Ev)_{j,l},
\end{aligned}
$$

for $j = 1, \ldots, n$. This representation has the advantage, that the second sum can be identified with the quasilinear operator $-\mathcal{A}(\mathcal{E}v)v$, i.e.

$$
\sum_{k,l,m=1}^{n} \mathcal{A}_{j,k}^{l,m}(\mathcal{E}(v))\partial_l\partial_m v_k = \sum_{l=1}^{n} \alpha(|\mathcal{E}(v)|^2)(\partial_l^2 v_j + \partial_l\partial_j v_l) + 4\alpha'(|\mathcal{E}(v)|^2)(\mathcal{E}(v) \colon \partial_l Ev)\mathcal{E}(v)_{j,l}.
$$

Combining this equation with $2\mathcal{E}(v) - 2Ev = B(I(v))^T\nabla v + (\nabla v)^T B(I(v))$ (see (4.12)), it follows that

$$
\begin{aligned}
&(\mathcal{A}(Ev_*)v)_j + (\operatorname{Div} 2\alpha(|\mathcal{E}(v)|^2)E(v))_j \\
&\quad = \sum_{k,l,m=1}^{n} (\mathcal{A}_{j,k}^{l,m}(\mathcal{E}(v)) - \mathcal{A}_{j,k}^{l,m}(Ev_*))\partial_l\partial_m v_k \\
&\qquad + 2\sum_{l=1}^{n} \alpha'(|\mathcal{E}(v)|^2)\big(\mathcal{E}(v) \colon \partial_l[B(I(v))^T\nabla v + (\nabla v)^T B(I(v))]\big)(Ev)_{j,l} \\
&\qquad - 2\sum_{l=1}^{n} \alpha'(|\mathcal{E}(v)|^2)\big(\mathcal{E}(v) \colon \partial_l Ev\big)\big(B(I(v))^T\nabla v + (\nabla v)^T B(I(v))\big)_{j,l}, \quad j = 1, \ldots, n.
\end{aligned}
$$

Finally, we write F_w in the form

$$
F_w(w) = \bar{F}_w(w + v_*), \tag{4.33}
$$

with

$$
\begin{aligned}
\bar{F}_w(v)_j := &\sum_{k,l,m=1}^{n} (\mathcal{A}_{j,k}^{l,m}(\mathcal{E}(v)) - \mathcal{A}_{j,k}^{l,m}(Ev_*))\partial_l\partial_m v_k \\
&+ 2\sum_{l=1}^{n} \alpha'(|\mathcal{E}(v)|^2)\big(\mathcal{E}(v)\colon \partial_l[B(I(v))^T\nabla v + (\nabla v)^T B(I(v))]\big)(Ev)_{j,l} \\
&- 2\sum_{l=1}^{n} \alpha'(|\mathcal{E}(v)|^2)\big(\mathcal{E}(v)\colon \partial_l Ev\big)\big(B(I(v))^T\nabla v + (\nabla v)^T B(I(v))\big)_{j,l} \\
&+ \bar{F}_j^{(1)}(v), \quad j = 1,\dots,n.
\end{aligned} \tag{4.34}
$$

Furthermore, we write the divergence condition (4.17) in the form

$$
\operatorname{div} w = F_d(w) \quad \text{in } (0,T_0)\times\Omega_0, \quad \text{with} \quad F_d(w) := \bar{F}_d(w + v_*), \tag{4.35}
$$

where $\bar{F}_d$ is defined in (4.18) as well as the transport equation (4.19) in the form

$$
\partial_t\zeta = G(w,\zeta) \quad \text{in } (0,T_0)\times\Omega_0, \quad \text{with} \quad G(w,\zeta) := \bar{G}(w + v_*, \tau_0 + \zeta), \tag{4.36}
$$

where $\bar{G}$ is defined in (4.20).

Last, we formulate the boundary condition (4.24), i.e.

$$
-(2\alpha(|Ev|^2)Ev - \theta)\nu_0 = \bar{H}(v,\theta,\eta) \quad \text{on } (0,T_0)\times\Gamma_{F,0},
$$

in terms of w, ψ, and ζ. By the definition $(v,\theta,\eta) = (w + v_*, \psi + \theta_*, \zeta + \tau_0)$, we rewrite (4.24) in the form

$$
\begin{aligned}
0 &= -(2\alpha(|E(w+v_*)|^2)E(w+v_*) - (\psi+\theta_*))\nu_0 - \bar{H}(w+v_*, \psi+\theta_*, \zeta+\tau_0) \\
&= -(2\alpha(|E(w+v_*)|^2)E(w+v_*) - (\psi+\theta_*))\nu_0 - \bar{H}(w+v_*, \psi+\theta_*, \zeta+\tau_0) \\
&= -\mathcal{B}_{\mathcal{N}}(Ev_*)(w,\psi) + h_* + H(w,\psi,\zeta),
\end{aligned}
$$

with

$$
h_* := 2(\alpha(0) - \alpha(|Ev_*|^2))Ev_*\nu_0 - h \in {}_0W_p^{\frac{1}{2}-\frac{1}{2p}}(0,T_0; L_p(\Gamma_{F,0})) \cap L_p(0,T_0; W_p^{1-\frac{1}{p}}(\Gamma_{F,0})), \tag{4.37}
$$

and

$$
\begin{aligned}
H(w,\psi,\zeta) := &\ \mathcal{B}_{\mathcal{N}}(Ev_*)(w,\psi) - h_* \\
&- 2\alpha(|E(w+v_*)|^2)E(w+v_*)\nu_0 + (\psi+\theta_*)\nu_0 - \bar{H}(w+v_*, \psi+\theta_*, \zeta+\tau_0),
\end{aligned} \tag{4.38}
$$

where $\bar{H}$ is defined in (4.25). Hence, we have

$$
\mathcal{B}_{\mathcal{N}}(Ev_*)(w,\psi) = h_* + H(w,\psi,\zeta) \quad \text{on } (0,T_0)\times\Gamma_{F,0}. \tag{4.39}
$$

We emphasize that $h_*(0) = 0$, by the definition of h (see (4.26)) and $v_*(0) = u_0$. Taking into account $h = \mu(\tau_0)\nu_0 + 2\alpha(0)Ev_*\nu_0 - \theta_*\nu_0$ (see (4.28)), we deduce that

$$
h_* = -2\alpha(|Ev_*|^2)Ev_*\nu_0 - \mu(\tau_0)\nu_0 + \theta_*\nu_0.
$$

Our next goal is to compute a different representation of H. We combine the previous representation of h_* and the representation of the Neumann boundary operator (1.5), i.e.

$$\mathcal{B}_{\mathcal{N}}(Ev_*)(w,\psi) = 2\alpha(|Ev_*|^2)Ew\nu_0 + 4\alpha'(|Ev_*|^2)(Ev_* : Ew)Ev_*\nu_0 - \psi\nu_0,$$

and obtain from (4.38)

$$\begin{aligned}
&H(w,\psi,\zeta) + \bar{H}(w+v_*,\psi+\theta_*,\zeta+\tau_0)\\
&\quad = \mathcal{B}_{\mathcal{N}}(Ev_*)(w,\psi) - h_* - 2\alpha(|E(w+v_*)|^2)E(w+v_*)\nu_0 + (\psi+\theta_*)\nu_0\\
&\quad = 2\alpha(|Ev_*|^2)E(w+v_*)\nu_0 + 4\alpha'(|Ev_*|^2)(Ev_* : Ew)Ev_*\nu_0 - (\psi+\theta_*)\nu_0 + \mu(\tau_0)\nu_0\\
&\qquad - 2\alpha(|E(w+v_*)|^2)E(w+v_*)\nu_0 + (\psi+\theta_*)\nu_0\\
&\quad = 2(\alpha(|Ev_*|^2) - \alpha(|E(w+v_*)|^2))E(w+v_*)\nu_0 + 4\alpha'(|Ev_*|^2)(Ev_* : Ew)Ev_*\nu_0 + \mu(\tau_0)\nu_0.
\end{aligned}$$

By the definition of $\bar{H}$ (see (4.25)), it follows that

$$\begin{aligned}
&H(w,\psi,\zeta)\\
&\quad = 2(\alpha(|Ev_*|^2) - \alpha(|E(w+v_*)|^2))E(w+v_*)\nu_0 + 4\alpha'(|Ev_*|^2)(Ev_* : Ew)Ev_*\nu_0 + \mu(\tau_0)\nu_0\\
&\qquad - \bar{H}(w+v_*,\psi+\theta_*,\zeta+\tau_0)\\
&\quad = H_w(w) + H_\psi(w,\psi) + H_\zeta(w,\zeta),
\end{aligned} \tag{4.40}$$

with

$$\begin{aligned}
H_w(w) &= 2(\alpha(|Ev_*|^2) - \alpha(|\mathcal{E}(w+v_*)|^2))E(w+v_*)\nu_0 + 4\alpha'(|Ev_*|^2)(Ev_* : Ew)Ev_*\nu_0\\
&\qquad + \bar{H}^{(1)}(w+v_*)\\
&= \bar{H}^{(1)}(w+v_*) + 2(\alpha(|Ev_*|^2) - \alpha(|\mathcal{E}(w+v_*)|^2))Ew\nu_0\\
&\qquad - 2\big(\alpha(|\mathcal{E}(w+v_*)|^2) - \alpha(|Ev_*|^2) - 2\alpha(|Ev_*|^2)(Ev_* : Ew)\big)Ev_*\nu_0,
\end{aligned} \tag{4.41}$$

as well as

$$H_\psi(w,\psi) := B(I(w+v_*))^T(\psi+\theta_*)\nu_0, \tag{4.42}$$

$$H_\zeta(w,\zeta) := -(\mu(\zeta+\tau_0) - \mu(\tau_0))\nu_0 - \mu(\zeta+\tau_0)B(I(w+v_*))^T\nu_0. \tag{4.43}$$

Summarizing, the system with zero initial values, which is equivalent to (4.4), reads

$$\left\{\begin{array}{rcll}
\rho\partial_t w + \mathcal{A}(Ev_*)w + \nabla\psi &=& f_* + F(w,\psi,\zeta) & \text{in } (0,T_0)\times\Omega_0,\\
\operatorname{div} w &=& F_d(w) & \text{in } (0,T_0)\times\Omega_0,\\
\partial_t\zeta &=& G(w,\zeta) & \text{in } (0,T_0)\times\Omega_0,\\
\mathcal{B}_{\mathcal{N}}(Ev_*)(w,\psi) &=& h_* + H(w,\psi,\zeta) & \text{on } (0,T_0)\times\Gamma_{F,0},\\
w &=& 0 & \text{on } (0,T_0)\times\Gamma_D,\\
w(0) &=& 0 & \text{in } \Omega_0,\\
\zeta(0) &=& 0 & \text{in } \Omega_0.
\end{array}\right. \tag{4.44}$$

We recall the definition of nonlinearities. The right-hand side of the generalized Stokes equation is given by (see (4.29) and (4.30))

$$f_* = -\alpha(0)\Delta v_* - \mathcal{A}(Ev_*)v_* + \operatorname{Div}\mu(\tau_0) \quad \text{and} \quad F(w,\psi,\zeta) = F_w(w) + F_\psi(w,\psi) + F_\zeta(w,\zeta),$$

with, see (4.14) and (4.31)–(4.34),

$$
\begin{aligned}
F_w(w) &= \Big(\sum_{k,l,m=1}^{n} (\mathcal{A}_{j,k}^{l,m}(\mathcal{E}(w+v_*)) - \mathcal{A}_{j,k}^{l,m}(Ev_*))\partial_l \partial_m (w+v_*)_k \Big)_{j=1,\dots,n} + \bar{F}_w^{(1)}(w+v_*),\\
F_\psi(w,\psi) &= -B(I(w+v_*))^T \nabla(\psi+\theta_*),\\
F_\zeta(w,\zeta) &= \operatorname{Div}(\mu(\zeta+\tau_0)-\mu(\tau_0)) + \Big(\sum_{k,l=1}^{n} B(I(w+v_*))_{l,k}\partial_l \mu(\zeta+\tau_0)_{j,k} \Big)_{j=1,\dots,n},
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{F}_w^{(1)}(v)_j = \operatorname{Div}\Big(&\alpha(|\mathcal{E}(v)|^2)\big(B(I(v))^T\nabla v + (\nabla v)^T B(I(v)) + \nabla v B(I(v))^T + (\nabla v)^T B(I(v))^T\\
&+ B(I(v))^T \nabla v B(I(v))^T + (\nabla v)^T B(I(v)) B(I(v))^T\big)\Big)_j\\
&+ 2\sum_{l=1}^{n} \alpha'(|\mathcal{E}(v)|^2)\big(\mathcal{E}(v) \colon \partial_l\big(B(I(v))^T\nabla v + (\nabla v)^T B(I(v))\big)\big)(Ev)_{j,l}\\
&- 2\sum_{l=1}^{n} \alpha'(|\mathcal{E}(v)|^2)\big(\mathcal{E}(v) \colon \partial_l Ev\big)\big(B(I(v))^T\nabla v + (\nabla v)^T B(I(v))\big)_{j,l}, \quad j=1,\dots,n.
\end{aligned}
$$

Further, the right-hand side of the divergence condition is given by, see (4.35),

$$
F_d(w) = -\operatorname{div}(B(I(w+v_*))(w+v_*)) = -(\nabla(w+v_*) : B(I(w+v_*))),
$$

and the right-hand side of the transport by, see (4.36),

$$
G(w,\zeta) = g(\nabla(w+v_*) + B(I(w+v_*))^T\nabla(w+v_*), \zeta+\tau_0).
$$

The right-hand side of the Neumann boundary condition on the boundary part $\Gamma_{F,0}$ reads, see (4.22), (4.26), and (4.39) –(4.43),

$$
\begin{aligned}
h_* = (\alpha(0) - \alpha(|Ev_*|^2))Ev_*\nu_0 - \mathcal{E}_t 2(\alpha(0)-\alpha(|Eu_0|^2))Eu_0\nu_0&\\
\in {}_0W_p^{\frac{1}{2}-\frac{1}{2p}}(0,T_0;L_p(\Gamma_{F,0})) \cap L_p(0,T_0;W_p^{1-\frac{1}{p}}(\Gamma_{F,0}))&,
\end{aligned}
$$

and

$$
H(w,\psi,\zeta) = H_w(w) + H_\psi(w,\psi) + H_\zeta(w,\zeta),
$$

where

$$
\begin{aligned}
H_w(w) &= -2(\alpha(|\mathcal{E}(w+v_*)|^2) - \alpha(|Ev_*|^2))Ew\nu_0\\
&\quad - 2\big(\alpha(|\mathcal{E}(w+v_*)|^2) - \alpha(|Ev_*|^2) - 2\alpha'(|Ev_*|^2)(Ev_* : Ew)\big)Ev_*\nu_0 + \bar{H}^{(1)}(w+v_*),\\
H_\psi(w,\psi) &= B(I(w+v_*))^T(\psi+\theta_*)\nu_0,\\
H_\zeta(w,\zeta) &= -(\mu(\zeta+\tau_0)-\mu(\tau_0))\nu_0 - \mu(\zeta+\tau_0)B(I(w+v_*))^T\nu_0,
\end{aligned}
$$

and

$$\bar{H}^{(1)}(v) = -\alpha(|\mathcal{E}(v)|^2)\big(B(I(v))^T\nabla v + (\nabla v)^T B(I(v)) + \nabla v B(I(v))^T + (\nabla v)^T B(I(v))^T + B(I(v))^T \nabla v B(I(v))^T + (\nabla v)^T B(I(v)) B(I(v))^T\big)\nu_0.$$

In order to shorten notation, we introduce the nonlinear term

$$N(w,\psi,\hat{\psi},\zeta) := (F(w,\psi,\zeta), F_d(w), G(w,\zeta), H(w,\hat{\psi},\zeta)),$$

and the function

$$n_* := (f_*, 0, 0, h_*).$$

From now on, we considered (4.44) instead of the equiavalent problem (4.4). The next theorem is equivalent to Theorem 4.1. □

Theorem 4.2. *Fix $n \in \mathbb{N}$, $n \geq 2$, $n+2 < p < \infty$, and $T_0, \rho > 0$. Let Ω_0 a domain with a compact boundary, such that the boundary $\partial\Omega_0 = \Gamma_{F,0} \cup \Gamma_D$ decomposes in two disjoint subsets $\Gamma_{F,0}$ and Γ_D, which are open and closed in $\partial\Omega_0$. Assume that $\alpha \in C^3([0,\infty))$, $\mu \in C^3(\mathbb{R}^{n\times n}, \mathbb{R}^{n\times n})$, and $g \in C^2(\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}, \mathbb{R}^{n\times n})$ satisfy the structure conditions*

$$\alpha(s) > 0, \quad \alpha(s) + 2s\alpha'(s) > 0, \quad s \geq 0 \quad \textit{and} \quad \mu(0) = g(0,0) = 0.$$

Then, for each $u_0 \in W_p^{2-\frac{2}{p}}(\Omega_0)$ and $\tau_0 \in H_p^1(\Omega_0)$, satisfying the compatibility conditions

$$\operatorname{div} u_0 = 0 \quad \textit{in } \Omega_0, \quad [2\alpha(|Eu_0|^2)Eu_0\nu_0 + \mu(\tau_0)\nu_0]_{tan} = 0 \quad \textit{on } \Gamma_{F,0}, \quad \textit{and} \quad u_0 = 0 \quad \textit{on } \Gamma_D,$$

there exists a time interval $(0,T)$ and a unique strong solution (w,ψ,ζ) of (4.44) on this time interval in the regularity class

$$\begin{array}{lcl} w & \in & {}_0H_p^1(0,T;L_p(\Omega_0)) \cap L_p(0,T;H_p^2(\Omega_0)), \\ \psi & \in & L_p(0,T;\widehat{H}_p^1(\Omega_0)), \\ \gamma_{\Gamma_{F_0}}\psi & \in & {}_0W_p^{\frac{1}{2}-\frac{1}{2p}}(0,T;L_p(\Gamma_{F,0})) \cap L_p(0,T;W_p^{1-\frac{1}{p}}(\Gamma_{F,0})), \\ \zeta & \in & {}_0W_\infty^1(0,T;L_p(\Omega_0)) \cap {}_0H_p^1(0,T;H_p^1(\Omega_0)). \end{array}$$

Proof of Theorem 4.2. Our first step is to formulate (4.44) in the form of a fixed point equation.

Fixed point formulation

We rewrite (4.44) as a fixed point problem in a suitable Banach space. For $n+2 < p < r < \infty$, we define the solution spaces

$${}_0\mathbb{E}(T) := \{(w,\psi,\hat{\psi},\zeta) \in {}_0\mathbb{E}_u(T,\Omega_0) \times \mathbb{E}_\pi(T,\Omega_0) \times {}_0\mathbb{E}_{\hat{\pi}}(T,\Gamma_{F_0}) \times {}_0\mathbb{E}_\tau(T,\Omega_0) : \gamma_{\Gamma_F}\psi = \hat{\psi},\ \gamma_{\Gamma_D} w = 0\},$$

where we used the same notation as in the previous chapters, i.e.

$$\begin{array}{lcl} {}_0\mathbb{E}_u(T,\Omega_0) & = & {}_0H_p^1(0,T;L_p(\Omega_0)) \cap L_p(0,T;H_p^2(\Omega_0)), \\ \mathbb{E}_\pi(T,\Omega_0) & = & L_p(0,T;\widehat{H}_p^1(\Omega_0)), \\ {}_0\mathbb{E}_{\hat{\pi}}(T,\Gamma_{F,0}) & = & {}_0W_p^{\frac{1}{2}-\frac{1}{2p}}(0,T;L_p(\Gamma_{F,0})) \cap L_p(0,T;W_p^{1-\frac{1}{p}}(\Gamma_{F,0})), \\ {}_0\mathbb{E}_\tau(T,\Omega_0) & = & {}_0H_r^1(0,T;L_p(\Omega_0)) \cap L_\infty(0,T;H_p^1(\Omega_0)). \end{array}$$

The data space is defined by

$$
{}_0\mathbb{F}(T) \;:=\; \mathbb{F}_f(T,\Omega_0) \times {}_0\mathbb{F}_d(T,\Omega_0,\Gamma_{F,0}) \times \mathbb{G}(T,\Omega_0) \times {}_0\mathbb{H}_u(T,\Gamma_{F,0}),
$$

where we also used the same notation as in the previous chapters, i.e.

$$
\begin{array}{lcl}
\mathbb{F}_f(T,\Omega_0) & = & L_p(0,T;L_p(\Omega_0)),\\
{}_0\mathbb{F}_d(T,\Omega_0,\Gamma_{F,0}) & = & {}_0H^1_p(0,T;{}^0\widehat{H}^{-1}_{p,\Gamma_{F,0}}(\Omega_0)) \cap L_p(0,T;H^1_p(\Omega_0)),\\
\mathbb{G}(T,\Omega_0) & = & L_r(0,T;L_p(\Omega_0)) \cap L_1(0,T;H^1_p(\Omega_0)),\\
{}_0\mathbb{H}_u(T,\Gamma) & = & {}_0W_p^{\frac{1}{2}-\frac{1}{2p}}(0,T;L_p(\Gamma_{F,0})) \cap L_p(0,T;W_p^{1-\frac{1}{p}}(\Gamma_{F,0})).
\end{array}
$$

Problem (4.44) can be rewritten as a fixed point problem of the map

$$
\Phi\colon {}_0\mathbb{E}(T) \to {}_0\mathbb{E}(T), \quad (w,\psi,\hat{\psi},\zeta) \mapsto \tilde{\Phi}_0(n_* + N(w,\psi,\hat{\psi},\zeta)),
$$

where

$$
\tilde{\Phi}_0\colon {}_0\mathbb{F}(T) \to {}_0\mathbb{E}(T), \quad (\tilde{f},\tilde{f}_d,\tilde{g},\tilde{h}) \mapsto (w,\psi,\hat{\psi},\zeta)
$$

denotes the solution operator to the following problem

$$
\left\{
\begin{array}{rcll}
\rho\partial_t w + \mathcal{A}(Ev_*)w + \nabla\psi & = & \tilde{f} & \text{in } (0,T_0)\times\Omega_0,\\
\operatorname{div} w & = & \tilde{f}_d & \text{in } (0,T_0)\times\Omega_0,\\
\partial_t\zeta & = & \tilde{g} & \text{in } (0,T_0)\times\Omega_0,\\
\mathcal{B}_{\mathcal{N}}(Ev_*)(w,\psi) & = & \tilde{h} & \text{on } (0,T_0)\times\Gamma_{F,0},\\
w & = & 0 & \text{on } (0,T_0)\times\Gamma_D,\\
w(0) & = & 0 & \text{in } \Omega_0,\\
\zeta(0) & = & 0 & \text{in } \Omega_0.
\end{array}
\right.
$$

The function ζ is explicitly given by

$$
\zeta(t) = \int_0^t \tilde{g}(s)ds, \quad t \in (0,T)
$$

and we obtain the estimates

$$
\|\zeta\|_{L_\infty(0,T;H^1_p(\Omega_0))} \le \|\tilde{g}\|_{L_1(0,T;H^1_p(\Omega_0))} \quad \text{and} \quad \|\partial_t\zeta\|_{T,\Omega_0,r,p} \le \|\tilde{g}\|_{T,\Omega_0,r,p}.
$$

Therefore, we infer

$$
\|\zeta\|_{\mathbb{E}_\tau(T,\Omega_0)} \le C\|\tilde{g}\|_{\mathbb{G}(T,\Omega_0)}. \tag{4.45}
$$

By the definition of the spaces ${}_0\mathbb{E}(T)$ and ${}_0\mathbb{F}(T)$, all compatibility conditions of Proposition 1.8 are fulfilled and hence, the mapping $\tilde{\Phi}_0$ is well defined. We have $n_* \in {}_0\mathbb{F}(T)$ and in Lemma 4.3, we show that $N\colon {}_0\mathbb{E}(T) \to {}_0\mathbb{F}(T)$. Hence, Φ is well-defined.

Mapping properties of the nonlinearities

The next subject is the nonlinearity N. The next lemma makes it legitimate to apply the contraction mapping principle to the fixed point problem $\Phi(z) = z$.

Lemma 4.3. *Fix $n \in \mathbb{N}$, $n \geq 2$, $n+2 < p < r < \infty$, and $T_0, R_0 > 0$. Let Ω_0 be a domain with a compact C^2-boundary and let $\Gamma_{F,0} \subset \partial\Omega_0$ be an open and closed subset of the boundary. Assume that $\alpha \in C^3([0,\infty))$, $\mu \in C^3(\mathbb{R}^{n\times n}, \mathbb{R}^{n\times n})$, and $g \in C^2(\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}, \mathbb{R}^{n\times n})$. Then*

$$N(w, \psi, \hat{\psi}, \zeta) \in {}_0\mathbb{F}(T_0), \quad (w, \psi, \hat{\psi}, \zeta) \in {}_0\mathbb{E}(T_0).$$

Furthermore, there exists a function $O\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $O(t) \to 0$ for $t \to 0$ and a constant C, such that for all $0 < T < T_0$, $0 < R < R_0$, and $(w, \psi, \hat{\psi}, \zeta), (w_j, \psi_j, \hat{\psi}_j, \zeta_j) \in \overline{B}_{{}_0\mathbb{E}(T)}(0, R)$, $j = 1, 2$, the estimates

$$\|N(w, \psi, \hat{\psi}, \zeta)\|_{{}_0\mathbb{F}(T)} \leq CR^2 + O(T), \quad \|G(w, \zeta)\|_{T,\Omega_0,\infty,p} + \|\nabla G(w, \zeta)\|_{T,\Omega_0,p,p} \leq C,$$

and

$$\begin{aligned}
&\|N(w_2, \psi_2, \hat{\psi}_2, \zeta_2) - N(w_1, \psi_1, \hat{\psi}_1, \zeta_1)\|_{{}_0\mathbb{F}(T)} \\
&\qquad\qquad \leq (CR + O(T))\|(w_2 - w_1, \psi_2 - \psi_1, \hat{\psi}_2 - \hat{\psi}_1, \zeta_2 - \zeta_1)\|_{{}_0\mathbb{E}(T)}
\end{aligned}$$

hold.

Before we give a proof of Lemma 4.3, we start with general results on the term $I(v)$ (the term $I(v)$ is defined in (4.8)) and on composition operators of the form $w \mapsto \Psi(I(w + v_*))$, where the function $\Psi\colon \mathbb{R}^{n\times n} \to \mathbb{R}$ is given. These results will be used frequently in the proof of Lemma 4.3.

Lemma 4.4. *Fix $n \in \mathbb{N}$, $n \geq 2$, $n+2 < p < \infty$, and $T_0 > 0$. Let Ω_0 be a domain with a compact C^2-boundary and $\Gamma \subset \partial\Omega_0$ be an open and closed subset of the boundary $\partial\Omega_0$.*

(a) *Then, there exists a constant $C > 0$, such that for all $T \in (0, T_0)$ and $v \in \mathbb{E}_u(T, \Omega_0)$ the estimates*

$$\begin{aligned}
\|I(v)\|_{L_\infty(0,T;H^1_p(\Omega_0))} &\leq T^{1-\frac{1}{p}}\|v\|_{L_p(0,T;H^2_p(\Omega_0))}, \\
\|\partial_t I(v)\|_{T,\Omega_0,p,p} &\leq \|v\|_{L_p(0,T;H^1_p(\Omega_0))}, \\
\|I(v)\|_{{}_0\mathbb{H}_u(T,\Gamma)} &\leq C(T\|v\|_{\mathbb{E}_u(T,\Omega_0)} + T^{1-\frac{1}{p}}\|v(0)\|_{\Omega_0,p})
\end{aligned}$$

hold.

(b) *Further, fix $\Psi \in C^3(\mathbb{R}^{n\times n}, \mathbb{R})$ with $\Psi(0) = 0$ and $R_0 > 0$. Then, there exists a function $O\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $O(t) \to 0$ for $t \to 0$, such that for all $0 < T < T_0$ and $w \in {}_0\mathbb{E}_u(T, \Omega_0)$ with $\|w\|_{\mathbb{E}_u(T,\Omega_0)} \leq R_0$ the estimates*

$$\begin{aligned}
\|\Psi(I(w + v_*))\|_{L_\infty(0,T;H^1_p(\Omega_0))} &\leq O(T), \\
\|D_w\Psi(I(w + v_*))\|_{\mathcal{L}({}_0\mathbb{E}_u(T,\Omega_0), L_\infty(0,T;H^1_p(\Omega_0)))} &\leq O(T), \\
\|\Psi(I(w + v_*))\|_{{}_0\mathbb{H}_u(T,\Gamma)} &\leq O(T), \\
\|D_w\Psi(I(w + v_*))\|_{\mathcal{L}({}_0\mathbb{E}_u(T,\Omega_0), {}_0\mathbb{H}_u(T,\Gamma))} &\leq O(T)
\end{aligned}$$

hold.

Proof. Let $0 < R_0$, $0 < T < T_0$, $v \in \mathbb{E}_u(T,\Omega_0)$, and $w \in {}_0\mathbb{E}_u(T,\Omega_0)$ with $\|w\|_{{}_0\mathbb{E}_u(T,\Omega_0)} \le R_0$. We denote by C a generic constant and by O a generic function with $O(t) \to 0$ for $t \to 0$, which may change from line to line, but are always independent of T, v, and w. First, we examine $I(v)$. For this purpose, we define

$$\tilde{I}(v)(t) = \int_0^t v(s)ds, \quad 0 < t < T.$$

We have $\nabla \tilde{I} = I$. By Hölder's inequality and $\|1\|_{t,p'}^p = t^{p-1}$ for $t \in (0,T)$ and $p' > 1$ with $\frac{1}{p} + \frac{1}{p'} = 1$, it holds that

$$\begin{aligned} \|\tilde{I}(v)(t)\|_{H_p^2(\Omega_0)} = \|\int_0^t v(s)ds\|_{H_p^2(\Omega_0)} &\le \int_0^t \|v(s)\|_{H_p^2(\Omega_0)}ds \le t^{1-\frac{1}{p}}\|v\|_{L_p(0,t;H_p^2(\Omega_0))} \\ &\le t^{1-\frac{1}{p}}\|v\|_{L_p(0,T;H_p^2(\Omega_0))}, \quad t \in (0,T). \end{aligned} \tag{4.46}$$

Taking the supremum over $t \in (0,T)$, we conclude that

$$\|\tilde{I}(v)\|_{L_\infty(0,T;H_p^2(\Omega_0))} \le T^{1-\frac{1}{p}}\|v\|_{L_p(0,T;H_p^2(\Omega_0))}.$$

This gives

$$\|I(v)\|_{L_\infty(0,T;H_p^1(\Omega_0))} \le \|\tilde{I}(v)\|_{L_\infty(0,T;H_p^2(\Omega_0))} \le T^{1-\frac{1}{p}}\|v\|_{L_p(0,T;H_p^2(\Omega_0))}.$$

This proves the first estimate in **(a)**. Integrating in (4.46) in time yields

$$\begin{aligned} \|\tilde{I}(v)\|_{L_p(0,T;H_p^2(\Omega_0))} = \Big(\int_0^T \|\tilde{I}(v)(t)\|_{H_p^2(\Omega_0)}^p dt\Big)^{\frac{1}{p}} &\le \Big(\int_0^T t^{p-1}dt\Big)^{\frac{1}{p}}\|v\|_{L_p(0,T;H_p^2(\Omega_0))} \\ &\le CT\|v\|_{L_p(0,T;H_p^2(\Omega_0))}. \end{aligned} \tag{4.47}$$

Since $\partial_t I(v)(t) = \nabla v(t)$, $t \in (0,T)$, it follows that $\|\partial_t I(v)\|_{T,\Omega_0,p,p} = \|\nabla v\|_{T,\Omega_0,p,p}$. This shows the second estimate in **(a)**. Moreover, we have

$$\partial_t \tilde{I}(v)(t) = v(t) = \int_0^t \partial_s v(s)ds + v(0), \quad t \in (0,T),$$

and consequently, the time derivative of $\tilde{I}(v)$ can be estimated by

$$\begin{aligned} \|\partial_t \tilde{I}(v)\|_{T,\Omega_0,p,p} &\le \Big(\int_0^T \Big(\int_0^t \|\partial_s v(s)\|_{\Omega_0,p}ds\Big)^p dt\Big)^{\frac{1}{p}} + T^{\frac{1}{p}}\|v(0)\|_{\Omega_0,p} \\ &\le \Big(\int_0^T t^{p-1}\|\partial_s v\|_{t,\Omega_0,p,p}^p dt\Big)^{\frac{1}{p}} + T^{\frac{1}{p}}\|v(0)\|_{\Omega_0,p} \\ &\le \Big(\int_0^T t^{p-1}dt\Big)^{\frac{1}{p}}\|\partial_s v\|_{T,\Omega_0,p,p} + T^{\frac{1}{p}}\|v(0)\|_{\Omega_0,p} \\ &\le C(T\|v\|_{H_p^1(0,T;L_p(\Omega_0))} + T^{\frac{1}{p}}\|v(0)\|_{\Omega_0,p}), \end{aligned}$$

where we once more used Hölder's inequality. Combining the previous equation with (4.47) and taking into account $\tilde{I}(v)(0) = 0$, we obtain

$$\|\tilde{I}(v)\|_{{}_0\mathbb{E}_u(T,\Omega_0)} \le C(T\|v\|_{\mathbb{E}_u(T,\Omega_0)} + T^{\frac{1}{p}}\|v(0)\|_{\Omega_0,p}).$$

By the continuity of the operator

$$\gamma_\Gamma \nabla \colon {}_0\mathbb{E}_u(T,\Omega_0) \to \mathbb{H}_u(T,\Gamma),$$

with uniform in T, $0<T<T_0$, estimate of the operator norm (see Proposition 1.15), it follows that

$$\|I(v)\|_{{}_0\mathbb{H}_u(T,\Gamma)} = \|\gamma_\Gamma \nabla \tilde{I}(v)\|_{{}_0\mathbb{H}_u(T,\Gamma)} \le C\|\tilde{I}(v)\|_{{}_0\mathbb{E}_u(T,\Omega_0)} \le C\big(T\|v\|_{\mathbb{E}_u(T,\Omega_0)} + T^{1-\frac{1}{p}}\|v(0)\|_{\Omega_0,p}\big).$$

This proves part **(a)**.

Next, we prove the claimed properties of the composition operator $w \mapsto \Psi(I(w+v_*))$. By **(a)** and Sobolev's embedding theorem, it holds that

$$\|I(w+v_*)\|_{T,\Omega_0,\infty,\infty} \le C. \tag{4.48}$$

On account of Proposition 1.19 and **(a)**, it may be concluded that

$$\begin{aligned}\|\Psi(I(w+v_*))\|_{L_\infty(0,T;H^1_p(\Omega_0))} &= \|\Psi(I(w+v_*)) - \Psi(0)\|_{L_\infty(0,T;H^1_p(\Omega_0))}\\ &\le C\|I(w+v_*)\|_{L_\infty(0,T;H^1_p(\Omega_0))}\\ &\le O(T).\end{aligned}$$

According to the chain rule and the proposition on Nemytskij operators (Proposition 1.17), we deduce that

$$\Psi(I(\cdot+v_*)) \in C^1({}_0\mathbb{E}_u(T,\Omega_0), L_\infty(0,T;H^1_p(\Omega_0)))$$

with

$$D_w\Psi(I(w+v_*))[\bar{w}] = \Psi'(I(w+v_*))I(\bar{w}), \quad \bar{w} \in {}_0\mathbb{E}_u(T,\Omega_0).$$

Due to (4.48) and **(a)**, it follows that

$$\begin{aligned}&\|D_w\Psi(I(w+v_*))[\bar{w}]\|_{L_\infty(0,T;H^1_p(\Omega_0))}\\ &\quad \le C(\|\Psi'(I(w+v_*))I(\bar{w})\|_{T,\Omega_0,\infty,p} + \|\nabla(\Psi'(I(w+v_*))I(\bar{w}))\|_{T,\Omega_0,\infty,p})\\ &\quad \le C(\|(\nabla\Psi)(I(w+v_*))\|_{T,\Omega_0,\infty,\infty} + \|(\nabla^2\Psi)(I(w+v_*))\|_{T,\Omega_0,\infty,\infty}\|I(w+v_*)\|_{L_\infty(0,T;H^1_p(\Omega_0))})\times\\ &\qquad \times \|I(\bar{w})\|_{L_\infty(0,T;H^1_p(\Omega_0))}\\ &\quad \le O(T)\|\bar{w}\|_{{}_0\mathbb{E}_u(T,\Omega_0)}, \quad \bar{w} \in {}_0\mathbb{E}(T,\Omega_0).\end{aligned}$$

Furthermore, by **(a)**, we infer

$$\|I(w+v_*)\|_{T,\Gamma,\infty,\infty} + \|I(w+v_*)\|_{{}_0\mathbb{H}_u(T,\Gamma)} \le C. \tag{4.49}$$

According to **(a)**, Proposition 1.19, and $I(w+v_*)(0)=0$, we have

$$\|\Psi(I(w+v_*))\|_{{}_0\mathbb{H}_u(T,\Gamma)} = \|\Psi(I(w+v_*)) - \Psi(0)\|_{{}_0\mathbb{H}_u(T,\Gamma)} \le C\|I(w+v_*)\|_{{}_0\mathbb{H}_u(T,\Gamma)} \le O(T).$$

In addition, by the proposition on Nemytskij operators and the chain rule, we deduce that

$$\Psi(I(\cdot+v_*)) \in C^1({}_0\mathbb{E}_u(T,\Omega_0), {}_0\mathbb{H}_u(T,\Gamma))$$

with

$$D_w\Psi(I(w+v_*))[\bar{w}] = \Psi'(I(w+v_*))I(\bar{w}), \quad \bar{w} \in {}_0\mathbb{E}_u(T,\Omega_0).$$

Therefore

$$\begin{aligned}\|D_w\Psi(I(w+v_*))[\bar{w}]\|_{{}_0\mathbb{H}_u(T,\Gamma)} &\le \|\Psi'(I(w+v_*))\|_{\mathbb{H}_u^\infty(T,\Gamma)}\|I(\bar{w})\|_{{}_0\mathbb{H}_u(T,\Gamma)} \\ &\le O(T)\|\bar{w}\|_{{}_0\mathbb{E}_u(T,\Omega_0)}, \quad \bar{w} \in {}_0\mathbb{E}_u(T,\Omega_0)\end{aligned}$$

follows from proposition on pointwise multiplications (Proposition 1.16), the proposition on Nemytskij operators, and **(a)**. □

Next, we prove Lemma 4.3.

Proof of Lemma 4.3. Fix $R_0, T_0 > 0$. Let $0 < R < R_0$, $0 < T < T_0$, and

$$(w,\psi,\hat{\psi},\zeta), (w_j,\psi_j,\hat{\psi}_j,\zeta_j) \in \overline{B}_{{}_0\mathbb{E}(T)}(0,R), \quad j = 1,2.$$

We denote by C a generic constant and by O a generic function with $O(t) \to 0$ for $t \to 0$, which may change from line to line, but are always independent of T, R, $(w,\psi,\hat{\psi},\zeta)$, as well as $(w_j,\psi_j,\hat{\psi}_j,\zeta_j)$. For short notation, we defined the abbreviations

$$(v,\theta,\hat{\theta},\eta) := (w+v_*, \psi+\theta_*, \hat{\psi}+\theta_*, \zeta+\tau_0).$$

On account of the proposition on embedding theorems (Proposition 1.14), there exists a constant C with

$$\begin{aligned}&\|w\|_{L_\infty(0,T;W^1_\infty(\Omega_0))} + \|w\|_{L_\infty(0,T;H^1_p(\Omega_0))} + \|v_*\|_{L_\infty(0,T;W^1_\infty(\Omega_0))} + \|v_*\|_{L_\infty(0,T;H^1_p(\Omega_0))} \\ &\quad + \|\zeta\|_{T,\Omega_0,\infty,\infty} + \|\tau_0\|_{\Omega_0,\infty} \le C\end{aligned} \tag{4.50}$$

as well as

$$\|w\|_{L_\infty(0,T;W^1_\infty(\Gamma_{F,0}))} + \|w\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} + \|v_*\|_{L_\infty(0,T;W^1_\infty(\Gamma_{F,0}))} + \|v_*\|_{\mathbb{H}_u(T,\Gamma_{F,0})} \le C. \tag{4.51}$$

Further, for short notation, we write for a function J

$$J(v,\theta,\hat{\theta},\eta) = J(w+v_*, \psi+\theta_*, \hat{\psi}+\theta_*, \zeta+\tau_0).$$

and for the Fréchet derivative

$$D_zJ(v,\theta,\hat{\theta},\eta) = D_zJ(w+v_*, \psi+\theta_*, \hat{\psi}+\theta_*, \zeta+\tau_0), \quad z \in \{w,\psi,\hat{\psi},\zeta\}.$$

We recall the definition

$$N(w,\psi,\hat{\psi},\zeta) = (F(w,\psi,\zeta), F_d(w), G(w,\zeta), H(w,\hat{\psi},\zeta))$$

and the definitions of F, F_d, G, and H are recalled below. We analyse each component of N separately.

Analysis of $F(w,\psi,\zeta)$

Our aim is to show that

$$\|F(w,\psi,\zeta)\|_{T,\Omega_0,p,p} \le CR^2 + O(T)$$

as well as

$$\begin{aligned}
&\|F(w_2,\psi_2,\zeta_2) - F(w_1,\psi_1,\zeta_1)\|_{T,\Omega_0,p,p} \\
&\qquad \le (CR + O(T))\|(w_2-w_1,\psi_2-\psi_1,\zeta_2-\zeta_1)\|_{{}_0\mathbb{E}_u(T,\Omega_0)\times\mathbb{E}_\pi(T,\Omega_0)\times{}_0\mathbb{E}_\tau(T,\Omega_0)}.
\end{aligned}$$

To show the second estimate, we prove an estimate of the Fréchet derivative of F of the form

$$\|DF(w,\psi,\zeta)\|_{\mathcal{L}({}_0\mathbb{E}_u(T,\Omega_0)\times\mathbb{E}_\pi(T,\Omega_0)\times{}_0\mathbb{E}_\tau(T,\Omega_0),L_p(0,T;L_p(\Omega_0)))} \le CR + O(T)$$

and apply the mean value theorem.

The nonlinearity F can be decomposed in a sum (see (4.30))

$$F(w,\psi,\zeta) = F_w(w) + F_\psi(w,\psi) + F_\zeta(w,\zeta).$$

We analyse each summand separately.
$\mathbf{F_w}$: We recall the definition of F_w. The term F_w can be decomposed in a sum (see (4.14) and (4.34))

$$F_w(w) = F_w^{(1)}(w) + F_w^{(2)}(w),$$

with

(4.52)
$$\begin{aligned}
F_w^{(1)}(w)_j &= \sum_{k,l,m=1}^{n} (\mathcal{A}_{j,k}^{l,m}(\mathcal{E}(v)) - \mathcal{A}_{j,k}^{l,m}(Ev_*))\partial_l\partial_m v_k, \quad j=1,\dots,n,\\
F_w^{(2)}(w)_j &= \operatorname{Div}\Big(\alpha(|\mathcal{E}(v)|^2)\big(B(I(v))^T\nabla v + (\nabla v)^T B(I(v)) + \nabla v B(I(v))^T + (\nabla v)^T B(I(v))^T\\
&\quad + B(I(v))^T \nabla v B(I(v))^T + (\nabla v)^T B(I(v)) B(I(v))^T\big)\Big)_j\\
&\quad + 2\sum_{l=1}^{n} \alpha'(|\mathcal{E}(v)|^2)\big(\mathcal{E}(v)\colon \partial_l(B(I(v))^T\nabla v + \nabla v B(I(v))^T)\big)(Ev)_{j,l}\\
&\quad - 2\sum_{l=1}^{n} \alpha'(|\mathcal{E}(v)|^2)\big(\mathcal{E}(v)\colon \partial_l Ev\big)\big(B(I(v))^T\nabla v + \nabla v B(I(v))^T\big)_{j,l}, \quad j=1,\dots,n.
\end{aligned}$$

Here, we used the convention to write $J(v)$ instead of $J(w+v_*)$ for a function J in order to shorten notation. We recall the definition of the transformed symmetric part of the gradient (see (4.12))

$$\mathcal{E}(v) = Ev + \frac{1}{2}\big(B(I(v))^T\nabla v + (\nabla v)^T B(I(v))\big).$$

By Lemma 4.4 and the proposition on embedding theorems (Proposition 1.14), we deduce that

$$\nabla v, B(I(v)) \in L_\infty(0,T;L_\infty(\Omega_0))$$

and thus, taking into account (4.50), we conclude that

$$\mathcal{E}(v) \in L_\infty(0,T;L_\infty(\Omega_0)) \quad \text{and} \quad \|\mathcal{E}(v)\|_{T,\Omega_0,\infty,\infty} \leq C. \tag{4.53}$$

The nonlinearities $F_w^{(1)}$ and $F_w^{(2)}$ are investigated separately. Due to (4.53) and the definition of $\mathbb{E}_u(T,\Omega_0)$, we have

$$(\mathcal{A}_{j,k}^{l,m}(\mathcal{E}(v)) - \mathcal{A}_{j,k}^{l,m}(Ev_*))_{j,k,l,m=1,\ldots,n} \in L_\infty(0,T;L_\infty(\Omega_0)) \quad \text{and} \quad \nabla^2 v \in L_p(0,T;L_p(\Omega_0)),$$

and hence

$$F_w^{(1)}(w) \in L_p(0,T;L_p(\Omega_0)). \tag{4.54}$$

Next, we estimate $\|F_w^{(1)}(w)\|_{T,\Omega_0,p,p}$. We have

$$\begin{aligned} &\|(\mathcal{A}_{j,k}^{l,m}(\mathcal{E}(v)) - \mathcal{A}_{j,k}^{l,m}(Ev_*))\partial_l\partial_m v_k\|_{T,\Omega_0,p,p} \\ &\qquad \leq \|(\mathcal{A}_{j,k}^{l,m}(\mathcal{E}(v)) - \mathcal{A}_{j,k}^{l,m}(Ev_*))\|_{T,\Omega_0,\infty,\infty}\|\partial_l\partial_m v_k\|_{T,\Omega_0,p,p}, \quad j,k,l,m = 1,\ldots,n. \end{aligned} \tag{4.55}$$

By the mean value theorem, the proposition on embedding theorems, Lemma 4.4, and (4.50), it follows that

$$\begin{aligned} &\|(\mathcal{A}_{j,k}^{l,m}(\mathcal{E}(v)) - \mathcal{A}_{j,k}^{l,m}(Ev_*))\|_{T,\Omega_0,\infty,\infty} \\ &\quad \leq C\|\mathcal{E}(v) - Ev_*\|_{T,\Omega_0,\infty,\infty} \\ &\quad \leq C(\|Ew\|_{T,\Omega_0,\infty,\infty} + \|B(I(v))\|_{T,\Omega_0,\infty,\infty}\|\nabla v\|_{T,\Omega_0,\infty,\infty}) \\ &\quad \leq C(\|w\|_{{}_0\mathbb{E}_u(T,\Omega_0)} + \|B(I(v))\|_{T,\Omega_0,\infty,\infty}\|\nabla v\|_{T,\Omega_0,\infty,\infty}) \\ &\quad \leq CR + O(T), \quad j,k,l,m = 1,\ldots,n. \end{aligned} \tag{4.56}$$

Combining the previous two inequalities, we deduce that

$$\|F_w^{(1)}(w)\|_{T,\Omega_0,p,p} \leq CR^2 + O(T).$$

The Fréchet derivative $DF_w^{(1)}(w)$ is our next subject. Due to the chain and product rule as well as the proposition on Nemytskij operators (Proposition 1.17), we infer

$$\begin{aligned} &D_w\big((\mathcal{A}_{j,k}^{l,m}(\mathcal{E}(v)) - \mathcal{A}_{j,k}^{l,m}(Ev_*))\partial_l\partial_m v_k\big)[\bar{w}] \\ &\quad = \sum_{\lambda_1,\lambda_2=1}^{n} (\partial_{(\lambda_1,\lambda_2)}\mathcal{A}_{j,k}^{l,m})(\mathcal{E}(v))D_w\mathcal{E}(v)_{\lambda_1,\lambda_2}[\bar{w}]\partial_l\partial_m v_k \\ &\qquad + (\mathcal{A}_{j,k}^{l,m}(\mathcal{E}(v)) - \mathcal{A}_{j,k}^{l,m}(Ev_*))\partial_l\partial_m \bar{w}_k, \quad \bar{w} \in {}_0\mathbb{E}_u(T,\Omega_0), \quad j,k,l,m = 1,\ldots,n. \end{aligned} \tag{4.57}$$

To estimate this formula, we investigate $D_w\mathcal{E}(v)_{\lambda_1,\lambda_2}[\bar{w}]$, $\lambda_1,\lambda_2 = 1,\ldots,n$, first. The product rule, the proposition on embedding theorems, Lemma 4.4, (4.50), and (4.53) lead to

$$\begin{aligned} &\|D_w\mathcal{E}(v)_{\lambda_1,\lambda_2}[\bar{w}]\|_{T,\Omega_0,\infty,\infty} \\ &\quad \leq \|E\bar{w}\|_{T,\Omega_0,\infty,\infty} \\ &\qquad + C(\|D_wB(I(v))[\bar{w}]\|_{T,\Omega_0,\infty,\infty}\|\nabla v\|_{T,\Omega_0,\infty,\infty} + \|B(I(v))\|_{T,\Omega_0,\infty,\infty}\|\nabla\bar{w}\|_{T,\Omega_0,\infty,\infty}) \\ &\quad \leq C\|\bar{w}\|_{{}_0\mathbb{E}_u(T,\Omega_0)}, \quad \bar{w} \in {}_0\mathbb{E}_u(T,\Omega_0), \quad \lambda_1,\lambda_2 = 1,\ldots,n. \end{aligned} \tag{4.58}$$

Combining the previous two inequalities and taking into account (4.53) and (4.56), we obtain

$$\begin{aligned}
&\|D_w\big((\mathcal{A}^{l,m}_{j,k}(\mathcal{E}(v)) - \mathcal{A}^{l,m}_{j,k}(Ev_*))\partial_l\partial_m v_k\big)[\bar{w}]\|_{T,\Omega_0,p,p} \\
&\quad \le C(\|(\nabla\mathcal{A}^{l,m}_{j,k})(\mathcal{E}(v))\|_{T,\Omega_0,\infty,\infty}\|D_w\mathcal{E}(v)[\bar{w}]\|_{T,\Omega_0,\infty,\infty}\|\partial_l\partial_m v_k\|_{T,\Omega_0,p,p} \\
&\qquad + \|(\mathcal{A}^{l,m}_{j,k}(\mathcal{E}(v)) - \mathcal{A}^{l,m}_{j,k}(Ev_*))\|_{T,\Omega_0,\infty,\infty}\|\partial_l\partial_m \bar{w}_k\|_{T,\Omega_0,p,p}) \\
&\quad \le (\|D_w\mathcal{E}(v)[\bar{w}]\|_{T,\Omega_0,\infty,\infty} + \|\partial_l\partial_m\bar{w}_k\|_{T,\Omega_0,p,p})(CR + O(T)) \\
&\quad \le (CR + O(T))\|\bar{w}\|_{{}_0\mathbb{E}_u(T,\Omega_0)}, \quad \bar{w} \in {}_0\mathbb{E}_u(T,\Omega_0), \quad j,k,l,m = 1,\dots,n.
\end{aligned}$$

Hence, we obtain

$$\|DF^{(1)}_w(w)\|_{\mathcal{L}({}_0\mathbb{E}_u(T,\Omega_0),L_p(0,T;L_p(\Omega_0)))} \le CR + O(T). \tag{4.59}$$

Now, we turn to $F^{(2)}_w$. We check that each component of $F^{(2)}_w$ is a sum, where every summand is of one of the forms

$$\begin{aligned}
F^{(2,1)}_w(w) &:= \alpha(|\mathcal{E}(v)|^2)(\partial_j b(I(v)))(\partial_{k_1} v_{k_2}), \\
F^{(2,2)}_w(w) &:= \alpha(|\mathcal{E}(v)|^2) b(I(v))(\partial_{j_1}\partial_{j_2} v_{j_3}), \\
F^{(2,3)}_w(w) &:= \alpha'(|\mathcal{E}(v)|^2) b(I(v))\mathcal{E}(v)_{j_1,j_2}(\partial_k\mathcal{E}(v)_{l_1,l_2})(\partial_{m_1} v_{m_2}), \\
F^{(2,4)}_w(w) &:= \alpha'(|\mathcal{E}(v)|^2)(\partial_j b(I(v)))\mathcal{E}(v)_{k_1,k_2}(\partial_{l_1} v_{l_2})(\partial_{m_1} v_{m_2}), \\
F^{(2,5)}_w(w) &:= \alpha'(|\mathcal{E}(v)|^2) b(I(v))\mathcal{E}(v)_{j_1,j_2}(\partial_{k_1}\partial_{k_2} v_{k_3})(\partial_{l_1} v_{l_2}),
\end{aligned}$$

where $b\colon \mathbb{R}^{n\times n} \to \mathbb{R}$ is a smooth function with $b(0) = 0$ and $j_\lambda, k_\lambda, l_\lambda, m_\lambda \in \{1,\dots,n\}$, $\lambda = 1,\dots,3$, are indices. It should be noted, that the function b is either of the form B_{j_1,j_2} or of the form $B_{j_1,j_2}B_{j_3,j_4}$ for suitable indices $j_1,\dots,j_4 \in \{1,\dots,n\}$.

First, we analyse the term $F^{(2,3)}_w$. In the definition of $F^{(2,3)}_w$, a term of the form $\nabla\mathcal{E}(v)$ appears. By Lemma 4.4 and (4.50), we deduce that

$$\begin{aligned}
&\|\nabla\mathcal{E}(v)\|_{T,\Omega_0,p,p} \\
&\quad \le \|\nabla^2 v\|_{T,\Omega_0,p,p} \\
&\qquad + C(\|\nabla^2 v\|_{T,\Omega_0,p,p}\|B(I(v))\|_{T,\Omega_0,\infty,\infty} + \|\nabla v\|_{T,\Omega_0,\infty,\infty}\|\nabla B(I(v))\|_{T,\Omega_0,p,p}) \\
&\quad \le CR + O(T).
\end{aligned} \tag{4.60}$$

Taking into account (4.53), we have $\|\alpha'(|\mathcal{E}(v)|^2)\|_{T,\Omega_0,\infty,\infty} \le C$. Hence, applying Lemma 4.4 and (4.50) once more, we obtain

$$\begin{aligned}
\|F^{(2,3)}_w(w)\|_{T,\Omega_0,p,p} &\le \|\alpha'(|\mathcal{E}(v)|^2)\|_{T,\Omega_0,\infty,\infty}\|b(I(v))\|_{T,\Omega_0,\infty,\infty}\|\mathcal{E}(v)_{j_1,j_2}\|_{T,\Omega_0,\infty,\infty}\times \\
&\quad \times \|\partial_k\mathcal{E}(v)_{l_1,l_2}\|_{T,\Omega_0,p,p}\|\partial_{m_1} v_{m_2}\|_{T,\Omega_0,\infty,\infty} \\
&\le O(T).
\end{aligned}$$

We estimate the remaining terms $F^{(2,1)}_w$, $F^{(2,2)}_w$, $F^{(2,4)}_w$, and $F^{(2,5)}_w$ in a similar way. Due to Lemma 4.4, (4.50), (4.53), and (4.60), we infer

$$X_0 := \{\alpha(|\mathcal{E}(v)|^2), \alpha'(|\mathcal{E}(v)|^2), \nabla v, \mathcal{E}(v), b(I(v))\} \subset L_\infty(0,T;L_\infty(\Omega_0))$$

and

$$X_1 := \{\nabla^2 v, \nabla \mathcal{E}(v), \nabla b(I(v))\} \subset L_p(0,T;L_p(\Omega_0)).$$

Each term $F_w^{2,1}, \ldots, F_w^{2,5}$ is product of terms in $X_0 \cup X_1$, where one and only one term belongs to X_1 and several terms belong to X_0. This shows $F_w^{(2)}(w) \in L_p(0,T;L_p(\Omega_0))$. Next, we estimate each element of $X_0 \cup X_1$. By Lemma 4.4, (4.50), (4.53), and (4.60), it follows that

$$\begin{aligned}\|\alpha(|\mathcal{E}(v)|^2)\|_{T,\Omega_0,\infty,\infty}, \|\alpha'(|\mathcal{E}(v)|^2)\|_{T,\Omega_0,\infty,\infty}, \|\alpha''(|\mathcal{E}(v)|^2)\|_{T,\Omega_0,\infty,\infty},& \\ \|\nabla v\|_{T,\Omega_0,\infty,\infty}, \|\mathcal{E}(v)\|_{T,\Omega_0,\infty,\infty} \le C,&\end{aligned} \tag{4.61}$$

and

$$\|b(I(v))\|_{T,\Omega_0,\infty,\infty}, \|\nabla b(I(v))\|_{T,\Omega_0,p,p} \le O(T), \tag{4.62}$$

as well as

$$\|\nabla^2 v\|_{T,\Omega_0,p,p}, \|\nabla \mathcal{E}(v)\|_{T,\Omega_0,p,p} \le CR + O(T). \tag{4.63}$$

Since at least one factor of $F_w^{2,1}, \ldots, F_w^{2,5}$ is bounded by $O(T)$ in the corresponding norm, it follows that

$$\sum_{j=1}^{5} \|F_w^{(2,j)}(w)\|_{T,\Omega_0,p,p} \le O(T)$$

and hence

$$\|F_w^{(2)}(w)\|_{T,\Omega_0,p,p} \le O(T). \tag{4.64}$$

Next, we analyse the Fréchet derivative $DF_w^{(2)}$. First, we investigate $DF_w^{(2,3)}$. By the product rule, we obtain

$$\begin{aligned} DF_w^{(2,3)}(w)[\bar{w}] &= (D_w \alpha'(|\mathcal{E}(v)|^2))[\bar{w}] b(I(v)) \mathcal{E}(v)_{j_1,j_2} (\partial_k \mathcal{E}(v)_{l_1,l_2})(\partial_{m_1} v_{m_2}) \\ &\quad + \alpha'(|\mathcal{E}(v)|^2)(D_w b(I(v)))[\bar{w}] \mathcal{E}(v)_{j_1,j_2} (\partial_k \mathcal{E}(v)_{l_1,l_2})(\partial_{m_1} v_{m_2}) \\ &\quad + \alpha'(|\mathcal{E}(v)|^2) b(I(v)) D_w \mathcal{E}(v)_{j_1,j_2}[\bar{w}] (\partial_k \mathcal{E}(v)_{l_1,l_2})(\partial_{m_1} v_{m_2}) \\ &\quad + \alpha'(|\mathcal{E}(v)|^2) b(I(v)) \mathcal{E}(v)_{j_1,j_2} (D_w(\partial_k \mathcal{E}(v)_{l_1,l_2}))[\bar{w}](\partial_{m_1} v_{m_2}) \\ &\quad + \alpha'(|\mathcal{E}(v)|^2) b(I(v)) \mathcal{E}(v)_{j_1,j_2} (\partial_k \mathcal{E}(v)_{l_1,l_2})(\partial_{m_1} \bar{w}_{m_2}), \quad \bar{w} \in {}_0\mathbb{E}_u(T,\Omega_0). \end{aligned} \tag{4.65}$$

First, we analyse the terms $D_w \alpha'(|\mathcal{E}(v)|^2)$ and $D_w(\partial_k \mathcal{E}(v)_{l_1,l_2})$, which appear in the first and forth line of (4.65). By the chain rule, the proposition on embedding theorems, the proposition on Nemytskij operators, (4.53), (4.58), and (4.61), it follows that

$$\begin{aligned} \|D_w \alpha'(\mathcal{E}(v))[\bar{w}]\|_{T,\Omega_0,\infty,\infty} &= \|2\alpha''(\mathcal{E}(v))(\mathcal{E}(v) : D_w \mathcal{E}(v)[\bar{w}])\|_{T,\Omega_0,\infty,\infty} \\ &\le C \|\alpha''(\mathcal{E}(v))\|_{T,\Omega_0,\infty,\infty} \|\mathcal{E}(v)\|_{T,\Omega_0,\infty,\infty} \|D_w \mathcal{E}(v)[\bar{w}]\|_{T,\Omega_0,\infty,\infty} \\ &\le C \|\bar{w}\|_{{}_0\mathbb{E}_u(T,\Omega_0)}, \quad \bar{w} \in {}_0\mathbb{E}_u(T,\Omega_0). \end{aligned} \tag{4.66}$$

Further, combining Lemma 4.4, the proposition on embedding theorems, (4.50), and (4.60) yields

$$
\begin{aligned}
&\|D_w \partial_k \mathcal{E}(v)_{l_1,l_2}[\bar{w}]\|_{T,\Omega_0,p,p} \\
&\quad = \|\partial_k D_w \mathcal{E}(v)_{l_1,l_2}[\bar{w}]\|_{T,\Omega_0,p,p} \\
&\quad = \|\partial_k E\bar{w}\|_{T,\Omega_0,p,p} + C\big(T^{\frac{1}{p}}\|\partial_k D_w B(I(v))[\bar{w}]\|_{T,\Omega_0,\infty,p}\|\nabla v\|_{T,\Omega_0,\infty,\infty} \\
&\qquad + \|D_w B(I(v))[\bar{w}]\|_{T,\Omega_0,\infty,\infty}\|\nabla^2 v\|_{T,\Omega_0,p,p} + T^{\frac{1}{p}}\|\partial_k B(I(v))\|_{T,\Omega_0,\infty,p}\|\nabla \bar{w}\|_{T,\Omega_0,\infty,\infty} \\
&\qquad + \|B(I(v))\|_{T,\Omega_0,\infty,\infty}\|\nabla^2 \bar{w}\|_{T,\Omega_0,p,p}\big) \\
&\quad \leq C\|\bar{w}\|_{{}_0\mathbb{E}_u(T,\Omega_0)}, \quad \bar{w} \in {}_0\mathbb{E}_u(T,\Omega_0).
\end{aligned} \tag{4.67}
$$

By the previous two estimates as well as Lemma 4.4, (4.58), (4.61)–(4.63), and (4.65), we infer

$$
\|DF_w^{(2,3)}(w)[\bar{w}]\|_{T,\Omega_0,p,p} \leq O(T)\|\bar{w}\|_{{}_0\mathbb{E}_u(T,\Omega_0)}, \quad \bar{w} \in {}_0\mathbb{E}_u(T,\Omega_0),
$$

and equivalently

$$
\|DF_w^{(2,3)}(w)\|_{\mathcal{L}({}_0\mathbb{E}_u(T,\Omega_0),L_p(0,T;L_p(\Omega_0)))} \leq O(T).
$$

We proceed the same way with the remaining terms $F_w^{(2,1)}$, $F_w^{(2,2)}$, $F_w^{(2,4)}$, and $F_w^{(2,5)}$. Each term $F_w^{(2,1)}, \ldots, F_w^{(2,5)}$ is a product with factors in $X_0 \cup X_1$. We estimate the Fréchet derivative of each element of $X_0 \cup X_1$. By (4.58), (4.66), and (4.67), we have

$$
\begin{aligned}
&\|D_w \alpha(|\mathcal{E}(v)|^2)\|_{\mathcal{L}({}_0\mathbb{E}(T,\Omega_0),L_\infty(0,T;L_\infty(\Omega_0)))}, \|D_w \alpha'(|\mathcal{E}(v)|^2)\|_{\mathcal{L}({}_0\mathbb{E}(T,\Omega_0),L_\infty(0,T;L_\infty(\Omega_0)))}, \\
&\quad \|D_w \nabla v\|_{\mathcal{L}({}_0\mathbb{E}(T,\Omega_0),L_\infty(0,T;L_\infty(\Omega_0)))}, \|D_w \mathcal{E}(v)\|_{\mathcal{L}({}_0\mathbb{E}(T,\Omega_0),L_\infty(0,T;L_\infty(\Omega_0)))}, \\
&\quad \|D_w \nabla^2 v\|_{\mathcal{L}({}_0\mathbb{E}(T,\Omega_0),L_p(0,T;L_p(\Omega_0)))}, \|D_w \nabla \mathcal{E}(v)\|_{\mathcal{L}({}_0\mathbb{E}(T,\Omega_0),L_p(0,T;L_p(\Omega_0)))} \leq C.
\end{aligned}
$$

Further, we get

$$
\|D_w b(I(v))\|_{\mathcal{L}({}_0\mathbb{E}(T,\Omega_0),L_\infty(0,T;L_\infty(\Omega_0)))}, \|D_w \nabla b(I(v))\|_{\mathcal{L}({}_0\mathbb{E}(T,\Omega_0),L_p(0,T;L_p(\Omega_0)))} \leq O(T),
$$

due to Lemma 4.4. Combining the previous two inequalities with (4.61)–(4.63), it follows that

$$
\sum_{j=1}^{5} \|D_w F_w^{(2,j)}(v)\|_{\mathcal{L}({}_0\mathbb{E}(T,\Omega_0),L_p(0,T;L_p(\Omega_0)))} \leq O(T),
$$

by the product rule, and hence

$$
\|DF_w^{(2)}(w)\|_{\mathcal{L}({}_0\mathbb{E}_u(T,\Omega_0),L_p(0,T;L_p(\Omega_0)))} \leq O(T).
$$

In summary, we showed

$$
\|F_w(w)\|_{T,\Omega_0,p,p} \leq CR^2 + O(T) \quad \text{and} \quad \|DF_w(w)\|_{\mathcal{L}({}_0\mathbb{E}_u(T,\Omega_0),L_p(0,T;L_p(\Omega_0)))} \leq O(T).
$$

$\mathbf{F}_\psi$: We proceed with the investigation of F_ψ. The term F_ψ is defined by (see (4.31))

$$
F_\psi(w,\psi) = -B(I(w+v_*))^T \nabla(\psi + \theta_*).
$$

By Lemma 4.4, it follows that

$$B(I(w+v_*))^T \in L_\infty(0,T;L_\infty(\Omega_0)).$$

Since $\nabla(\psi+\theta_*) \in L_p(0,T;L_p(\Omega_0))$, we have

$$F_\psi(w,\psi) \in L_p(0,T;L_p(\Omega_0)).$$

By Hölder's inequality and Lemma 4.4, we deduce that

$$\|F_\psi(w,\psi)\|_{T,\Omega_0,p,p} \leq O(T).$$

Next, we analyse the Fréchet derivative of F_ψ. We obtain

$$\begin{aligned}
&\|DF_\psi(w,\psi)[\bar{w},\bar{\psi}]\|_{T,\Omega_0,p,p}\\
&\quad = \|D_w B(I(w+v_*))[\bar{w}]\|_{T,\Omega_0,\infty,\infty}\|\nabla(\psi+\theta_*)\|_{T,\Omega_0,p,p} + \|B(I(w+v_*))\|_{T,\Omega_0,\infty,\infty}\|\nabla\bar{\psi}\|_{T,\Omega_0,p,p}\\
&\quad \leq O(T)(\|\bar{w}\|_{{}_0\mathbb{E}_u(T,\Omega_0)} + \|\bar{\psi}\|_{\mathbb{E}_\pi(T,\Omega_0)}), \quad (\bar{w},\bar{\psi}) \in {}_0\mathbb{E}_u(T,\Omega_0)\times\mathbb{E}_\pi(T,\Omega_0),
\end{aligned}$$

by the product rule and Lemma 4.4, and hence

$$\|DF_\psi(w,\psi)\|_{\mathcal{L}({}_0\mathbb{E}_u(T,\Omega_0)\times\mathbb{E}_\pi(T,\Omega_0),L_p(0,T;L_p(\Omega_0)))} \leq O(T).$$

$\mathbf{F}_\zeta$: Finally, we analyse F_ζ. The term F_ζ is defined by

$$F_\zeta(w,\zeta)_j = \operatorname{Div}(\mu(\zeta+\tau_0)-\mu(\tau_0))_j + \sum_{k,l=1}^n (\partial_l\mu(\zeta+\tau_0))_{j,k}B_{l,k}(I(w+v_*)), \quad j=1,\dots,n.$$

First, we investigate $\operatorname{Div}(\mu(\tau_0))$. By the chain rule, we have

$$\|\operatorname{Div}(\mu(\tau_0))\|_{T,\Omega_0,p,p} \leq C\|\nabla\tau_0\|_{T,\Omega_0,p,p} \leq CT^{\frac{1}{p}}\|\tau_0\|_{H^1_p(\Omega_0)} \leq O(T).$$

The remaining terms in F_ζ are of one of the forms

$$\begin{aligned}
F_\zeta^{(1)}(\zeta) &= (\partial_{(j_1,j_2)}\mu_{k_1,k_2})(\zeta+\tau_0)\partial_l(\zeta+\tau_0)_{m_1,m_2},\\
F_\zeta^{(2)}(w,\zeta) &= F_\zeta^{(1)}(\zeta)b(I(w+v_*)),
\end{aligned}$$

where $b\colon \mathbb{R}^{n\times n}\to\mathbb{R}$ is a smooth function with $b(0)=0$ and $j_1,j_2,k_1,k_2,l,m_1,m_2 \in \{1,\dots,n\}$. Due to (4.50), we have

$$\|F_\zeta^{(1)}(\zeta)\|_{T,\Omega_0,p,p} \leq C\|\nabla(\zeta+\tau_0)\|_{T,\Omega_0,p,p} \leq O(T), \tag{4.68}$$

and, taking into account Lemma 4.4, we deduce that

$$\|F_\zeta^{(2)}(w,\zeta)\|_{T,\Omega_0,p,p} \leq \|F_\zeta^{(1)}(\zeta)\|_{T,\Omega_0,p,p}\|b(I(w+v_*))\|_{T,\Omega_0,\infty,\infty} \leq O(T).$$

Combining the previous two inequalities, we have

$$\|F_\zeta(w,\zeta)\|_{T,\Omega_0,p,p} \leq O(T).$$

Second, we estimate the Fréchet derivative of $F^{(1)}_\zeta$ and $F^{(2)}_\zeta$. The product rule, Sobolev's embedding theorem, the proposition on Nemytskij operators, and (4.50) lead to

$$\begin{aligned}
&\|DF^{(1)}_\zeta(\zeta)[\bar\zeta]\|_{T,\Omega_0,p,p}\\
&\quad\le CT^{\frac{1}{p}}(\|(\nabla^2\mu)(\eta)\|_{T,\Omega_0,\infty,\infty}\|\bar\zeta\|_{T,\Omega_0,\infty,\infty}\|\nabla\eta\|_{T,\Omega_0,\infty,p}+\|(\nabla\mu)(\eta)\|_{T,\Omega_0,\infty,\infty}\|\nabla\bar\zeta\|_{T,\Omega_0,\infty,p})\\
&\quad\le CT^{\frac{1}{p}}\|\bar\zeta\|_{L_\infty(0,T;H^1_p(\Omega_0))},\quad (\bar w,\bar\zeta)\in {}_0\mathbb{E}_u(T,\Omega_0)\times{}_0\mathbb{E}_\zeta(T,\Omega_0)
\end{aligned}$$

and, on account of Lemma 4.4 and (4.68), we deduce that

$$\begin{aligned}
&\|DF^{(2)}_\zeta(w,\zeta)[\bar w,\bar\zeta]\|_{T,\Omega_0,p,p}\\
&\quad\le\|D_\zeta F^{(1)}_\zeta(\zeta)[\bar\zeta]\|_{T,\Omega_0,p,p}\|b(I(v))\|_{T,\Omega_0,\infty,\infty}+\|F^{(1)}_\zeta(\zeta)\|_{T,\Omega_0,p,p}\|D_w b(I(v))[\bar w]\|_{T,\Omega_0,\infty,\infty}\\
&\quad\le O(T)(\|\bar w\|_{{}_0\mathbb{E}_u(T,\Omega_0)}+\|\bar\zeta\|_{{}_0\mathbb{E}_\zeta(T,\Omega_0)}),\quad (\bar w,\bar\zeta)\in {}_0\mathbb{E}_u(T,\Omega_0)\times{}_0\mathbb{E}_\zeta(T,\Omega_0).
\end{aligned}$$

This implies

$$\|DF_\zeta(w,\zeta)\|_{\mathcal{L}({}_0\mathbb{E}_u(T,\Omega_0)\times{}_0\mathbb{E}_\zeta(T,\Omega_0),L_p(0,T;L_p(\Omega_0)))}\le O(T).$$

In summary, we have

$$\|F(w,\psi,\zeta)\|_{L_p(0,T;L_p(\Omega_0))}\le O(T)+CR^2$$

and

$$\|DF(w,\psi,\zeta)\|_{\mathcal{L}({}_0\mathbb{E}_u(T,\Omega_0)\times\mathbb{E}_\pi(T,\Omega_0)\times{}_0\mathbb{E}_\zeta(T,\Omega_0),L_p(0,T;L_p(\Omega_0)))}\le O(T)+CR.$$

By the mean value theorem, it follows that

$$\begin{aligned}
&\|F(w_2,\psi_2,\zeta_2)-F(w_1,\psi_1,\zeta_1)\|_{L_p(0,T;L_p(\Omega_0))}\\
&\qquad\le(O(T)+CR)\|(w_2-w_1,\psi_2-\psi_1,\zeta_2-\zeta_1)\|_{{}_0\mathbb{E}_u(T,\Omega_0)\times\mathbb{E}_\pi(T,\Omega_0)\times{}_0\mathbb{E}_\zeta(T,\Omega_0)}.
\end{aligned}$$

Analysis of $F_d(w)$

We recall the definition (see (4.35))

$$F_d(w)=-(\nabla(w+v_*):B(I(w+v_*))).$$

First, we analyse F_d in $L_p(0,T;H^1_p(\Omega_0))$. By the algebra property of $H^1_p(\Omega_0)$ and Lemma 4.4, we obtain

$$\begin{aligned}
\|F_d(w)\|_{L_p(0,T;H^1_p(\Omega_0))}&=\|(\nabla(w+v_*):B(I(w+v_*)))\|_{L_p(0,T;H^1_p(\Omega_0))}\\
&\le\|\nabla(w+v_*)\|_{L_p(0,T;H^1_p(\Omega_0))}\|B(I(w+v_*))\|_{L_\infty(0,T;H^1_p(\Omega_0))}\\
&\le O(T)
\end{aligned}$$

as well as

$$\begin{aligned}
&\|F_d(w_2)-F_d(w_1)\|_{L_p(0,T;H^1_p(\Omega_0))}\\
&\quad\le\|\nabla(w_2-w_1)\|_{L_p(0,T;H^1_p(\Omega_0))}\|B(I(w_2+v_*))\|_{L_\infty(0,T;H^1_p(\Omega_0))}\\
&\qquad+\|\nabla(w_1+v_*)\|_{L_p(0,T;H^1_p(\Omega_0))}\|B(I(w_2+v_*))-B(I(w_1+v_*))\|_{L_\infty(0,T;H^1_p(\Omega_0))}\\
&\quad\le O(T)\|w_2-w_1\|_{{}_0\mathbb{E}_u(T,\Omega_0)}
\end{aligned}$$

due to the mean value theorem. Second, we analyse $F_{F_d(w),0}$ in $H^1_p(0,T;{}^0\widehat{H}^{-1}_{p,\Gamma_{F,0}}(\Omega_0))$ (the expression $F_{F_d(w),0} \in H^1_p(0,T;{}^0\widehat{H}^{-1}_{p,\Gamma_{F,0}}(\Omega_0))$ is defined in (1.10)). For this purpose, we use the representation (see (4.35))

$$F_d(w) = -\operatorname{div}(B(I(w+v_*))(w+v_*)).$$

We compute a more explicit representation of $F_{F_d(w),0}$. For $\varphi \in \widehat{H}^1_{p',\Gamma_{F,0}}(\Omega_0)$, where $1 < p' < \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$, we see that

$$F_{F_d(w),0}(t)\varphi = \int_{\Omega_0} F_d(w)(t)\varphi = -\int_{\Omega_0} (\operatorname{div}(B(I(v)(t))v(t)))\varphi = \int_{\Omega_0} B(I(v))v(t)\cdot\nabla\varphi, \quad 0 < t < T,$$

since $\gamma_{\Gamma_{F,0}}\varphi = 0$ and $\gamma_{\Gamma_D} v = 0$, and hence the appearing boundary integrals vanish. It follows that

$$F_{F_d(w),0}\varphi = \int_{\Omega_0} B(I(v))v\cdot\nabla\varphi, \quad \varphi \in {}_0\widehat{H}^1_{p',\Gamma_{F,0}}(\Omega_0).$$

Hence, the investigation of $F_{F_d(w),0}$ in $H^1_p(0,T;{}^0\widehat{H}^{-1}_p(\Omega_0))$ reduces to the investigation of $B(I(v))v$ in $H^1_p(0,T;L_p(\Omega_0))$. By Shibata and Shimizu [SS07b, (2.19) and (2.51)], it holds that

$$\|B(I(w+v_*))(w+v_*)\|_{H^1_p(0,T;L_p(\Omega_0))} \leq O(T),$$

and

$$\|B(I(w_2+v_*))(w_2+v_*) - B(I(w_1+v_*))(w_1+v_*)\|_{H^1_p(0,T;L_p(\Omega_0))} \leq O(T)\|w_2-w_1\|_{{}_0\mathbb{E}_u(T,\Omega_0)},$$

and consequently

$$\|F_{F_d(w),0}\|_{H^1_p(0,T;{}^0\widehat{H}^{-1}_{p,\Gamma_{F,0}}(\Omega_0))} \leq C\|B(I(w+v_*))(w+v_*)\|_{H^1_p(0,T;L_p(\Omega_0))} \leq O(T),$$

and

$$\begin{aligned}
&\|F_{F_d(w_2),0} - F_{F_d(w_1),0}\|_{H^1_p(0,T;{}^0\widehat{H}^{-1}_{p,\Gamma_{F,0}}(\Omega_0))} \\
&\quad \leq C\|B(I(w_2+v_*))(w_2+v_*) - B(I(w_1+v_*))(w_1+v_*)\|_{H^1_p(0,T;L_p(\Omega_0))} \\
&\quad \leq O(T)\|w_2-w_1\|_{{}_0\mathbb{E}_u(T,\Omega_0)}.
\end{aligned}$$

In summary, we have

$$\|F_d(w)\|_{{}_0\mathbb{F}_d(T,\Omega_0,\Gamma_{F,0})} \leq O(T)\|w\|_{{}_0\mathbb{E}_u(T,\Omega_0)},$$

and

$$\|F_d(w_2) - F_d(w_1)\|_{{}_0\mathbb{F}_d(T,\Omega_0,\Gamma_{F,0})} \leq O(T)\|w_2-w_1\|_{{}_0\mathbb{E}_u(T,\Omega_0)}.$$

Analysis of $G(w,\zeta)$

We recall the definition (see (4.36))

$$G(w,\zeta) = g(\nabla v + B(I(v))^T \nabla v, \eta).$$

For short notation, we define

$$\mathcal{D}(v) = \nabla v + B(I(v))^T \nabla v.$$

Then, we can write $G(w,\zeta) = g(\mathcal{D}(v), \eta)$. In a first step, we analyse $\mathcal{D}(v)$. By the mean value theorem, the proposition on embedding theorems, Lemma 4.4, and (4.50), we have

$$\|\mathcal{D}(v)\|_{T,\Omega_0,\infty,\infty} \leq \|\nabla v\|_{T,\Omega_0,\infty,\infty} + \|B(I(v))\|_{T,\Omega_0,\infty,\infty} \|\nabla v\|_{T,\Omega_0,\infty,\infty} \leq C, \tag{4.69}$$

and

$$\begin{aligned} &\|\mathcal{D}(v_2) - \mathcal{D}(v_1)\|_{T,\Omega_0,\infty,\infty} \\ &\quad \leq \|\nabla(v_2 - v_1)\|_{T,\Omega_0,\infty,\infty} + \|B(I(v_2)) - B(I(v_1))\|_{T,\Omega_0,\infty,\infty} \|\nabla v_2\|_{T,\Omega_0,\infty,\infty} \\ &\qquad + \|B(I(v_1))\|_{T,\Omega_0,\infty,\infty} \|\nabla(v_2 - v_1)\|_{T,\Omega_0,\infty,\infty} \\ &\quad \leq C\|w_2 - w_1\|_{{}_0\mathbb{E}_u(T,\Omega_0)}, \end{aligned} \tag{4.70}$$

and

$$\begin{aligned} \|\mathcal{D}(v)\|_{T,\Omega_0,\infty,p} &\leq \|\nabla v\|_{T,\Omega_0,\infty,p} + \|B(I(v))^T \nabla v\|_{T,\Omega_0,\infty,p} \\ &\leq \|\nabla v\|_{T,\Omega_0,\infty,p} + \|B(I(v))\|_{T,\Omega_0,\infty,\infty} \|\nabla v\|_{T,\Omega_0,\infty,p} \\ &\leq C, \end{aligned} \tag{4.71}$$

and

$$\begin{aligned} &\|\mathcal{D}(v_2) - \mathcal{D}(v_1)\|_{T,\Omega_0,\infty,p} \\ &\quad \leq \|v_2 - v_1\|_{T,\Omega_0,\infty,p} + \|B(I(v_2)) - B(I(v_1))\|_{T,\Omega_0,\infty,\infty} \|\nabla v_2\|_{T,\Omega_0,\infty,p} \\ &\qquad + \|B(I(v_1))\|_{T,\Omega_0,\infty,\infty} \|\nabla(v_2 - v_1)\|_{T,\Omega_0,\infty,p} \\ &\quad \leq C\|w_2 - w_1\|_{{}_0\mathbb{E}_u(T,\Omega_0)}. \end{aligned} \tag{4.72}$$

Further, we infer

$$\begin{aligned} &\|\nabla \mathcal{D}(v)\|_{T,\Omega_0,p,p} \\ &\quad \leq \|\nabla^2 v\|_{T,\Omega_0,p,p} + \|\nabla B(I(v))\|_{T,\Omega_0,p,p} \|\nabla v\|_{T,\Omega_0,\infty,\infty} + \|B(I(v))\|_{T,\Omega_0,\infty,\infty} \|\nabla^2 v\|_{T,\Omega_0,p,p} \\ &\quad \leq CR + O(T) \end{aligned} \tag{4.73}$$

by the product rule, (4.50), and Lemma 4.4. By similar arguments, we deduce that

$$\begin{aligned} \|\nabla \mathcal{D}(v_2) - \nabla \mathcal{D}(v_1)\|_{T,\Omega_0,p,p} &\leq \|\nabla^2 (v_2 - v_1)\|_{T,\Omega_0,p,p} \\ &\quad + \|\nabla(B(I(v_2)) - B(I(v_1)))\|_{T,\Omega_0,p,p} \|\nabla v_2\|_{T,\Omega_0,\infty,\infty} \\ &\quad + \|\nabla B(I(v_1))\|_{T,\Omega_0,p,p} \|\nabla(v_2 - v_1)\|_{T,\Omega_0,\infty,\infty} \\ &\quad + \|B(I(v_2))\|_{T,\Omega_0,\infty,\infty} \|\nabla^2 (v_2 - v_1)\|_{T,\Omega_0,p,p} \\ &\quad + \|B(I(v_2)) - B(I(v_1))\|_{T,\Omega_0,\infty,\infty} \|\nabla^2 v_1\|_{T,\Omega_0,p,p} \\ &\leq C\|w_2 - w_1\|_{{}_0\mathbb{E}_u(T,\Omega_0)}. \end{aligned} \tag{4.74}$$

Fix $\tilde{r} \in \{1, r, \infty\}$. By the mean value theorem, $g(0,0) = 0$, (4.50), (4.69), (4.71), and (4.72), it follows that

$$\begin{aligned}\|G(w,\zeta)\|_{T,\Omega_0,\tilde{r},p} &\le \|g(\mathcal{D}(v),\eta) - g(0,0)\|_{T,\Omega_0,\tilde{r},p}\\ &\le C(\|\mathcal{D}(v)\|_{T,\Omega_0,\tilde{r},p} + \|\eta\|_{T,\Omega_0,\tilde{r},p})\\ &\le CT^{\frac{1}{\tilde{r}}}\end{aligned}$$

as well as

$$\begin{aligned}\|G(w_2,\zeta_2) - G(w_1,\zeta_1)\|_{T,\Omega_0,\tilde{r},p} &\le \|g(\mathcal{D}(v_2),\eta_2) - g(\mathcal{D}(v_1),\eta_1)\|_{T,\Omega_0,\tilde{r},p}\\ &\le C(\|\mathcal{D}(v_2) - \mathcal{D}(v_1)\|_{T,\Omega_0,\tilde{r},p} + \|\eta_2 - \eta_1\|_{T,\Omega_0,\tilde{r},p})\\ &\le CT^{\frac{1}{\tilde{r}}}(\|w_2 - w_1\|_{{}_0\mathbb{E}_u(T,\Omega_0)} + \|\zeta_2 - \zeta_1\|_{{}_0\mathbb{E}_\tau(T,\Omega_0)}).\end{aligned}$$

In particular, we have $\|G(w,\zeta)\|_{T,\Omega_0,\infty,p} \le C$. To analyse G in $\mathbb{G}(T,\Omega_0)$, it remains to show that

$$\|\nabla G(w,\zeta)\|_{T,\Omega_0,1,p} \le O(T),$$

and

$$\|\nabla(G(w_2,\zeta_2) - G(w_1,\zeta_1))\|_{T,\Omega_0,1,p} \le O(T)(\|w_2 - w_1\|_{{}_0\mathbb{E}_u(T,\Omega_0)} + \|\zeta_2 - \zeta_1\|_{{}_0\mathbb{E}_\zeta(T,\Omega_0)}).$$

Fix $\tilde{s} \in \{1, p\}$. By the chain rule, (4.69), and (4.73), it follows that

$$\begin{aligned}\|\nabla G(w,\zeta)\|_{T,\Omega_0,\tilde{s},p} = \|\nabla g(\mathcal{D}(v),\eta)\|_{T,\Omega_0,\tilde{s},p} &\le C(\|\nabla\mathcal{D}(v)\|_{T,\Omega_0,\tilde{s},p} + \|\nabla\eta\|_{T,\Omega_0,\tilde{s},p})\\ &\le CT^{\frac{1}{p}-\frac{1}{\tilde{s}}}\|\nabla\mathcal{D}(v)\|_{T,\Omega_0,p,p} + O(T)\|\eta\|_{T,\Omega_0,\infty,p}.\end{aligned}$$

In particular, we have $\|\nabla G(w,\zeta)\|_{T,\Omega_0,p,p} \le C$. Taking into account the mean value theorem, the proposition on embedding theorems, (4.69), (4.70), (4.73), and (4.74), we infer

$$\begin{aligned}&\|\nabla(G(w_2,\zeta_2) - G(w_1,\zeta_1))\|_{T,\Omega_0,1,p}\\ &\quad= \|\nabla(g(\mathcal{D}(v_2),\eta_2) - g(\mathcal{D}(v_2),\eta_2))\|_{T,\Omega_0,1,p}\\ &\quad\le \|(\nabla g)(\mathcal{D}(v_2),\eta_2) - (\nabla g)(\mathcal{D}(v_1),\eta_1)\|_{T,\Omega_0,\infty,\infty}(\|\nabla\mathcal{D}(v_2)\|_{T,\Omega_0,1,p} + \|\nabla\eta_2\|_{T,\Omega_0,1,p})\\ &\qquad+ \|(\nabla g)(\mathcal{D}(v_1),\eta_1)\|_{T,\Omega_0,\infty,\infty}(\|\nabla(\mathcal{D}(v_2) - \mathcal{D}(v_1))\|_{T,\Omega_0,1,p} + \|\nabla(\eta_2 - \eta_1)\|_{T,\Omega_0,1,p})\\ &\quad\le C(\|\mathcal{D}(v_2) - \mathcal{D}(v_1)\|_{T,\Omega_0,\infty,\infty} + \|\eta_2 - \eta_1\|_{T,\Omega_0,\infty,\infty})(\|\nabla\mathcal{D}(v_2)\|_{T,\Omega_0,1,p} + \|\nabla\eta_2\|_{T,\Omega_0,1,p})\\ &\qquad+ C(\|\nabla(\mathcal{D}(v_2) - \mathcal{D}(v_1))\|_{T,\Omega_0,1,p} + \|\nabla(\eta_2 - \eta_1)\|_{T,\Omega_0,1,p})\\ &\quad\le O(T)(\|\mathcal{D}(v_2) - \mathcal{D}(v_1)\|_{T,\Omega_0,\infty,\infty} + \|\eta_2 - \eta_1\|_{T,\Omega_0,\infty,\infty})(\|\nabla\mathcal{D}(v_2)\|_{T,\Omega_0,p,p} + \|\nabla\eta_2\|_{T,\Omega_0,\infty,p})\\ &\qquad+ O(T)(\|\nabla(\mathcal{D}(v_2) - \mathcal{D}(v_1))\|_{T,\Omega_0,p,p} + \|\nabla(\eta_2 - \eta_1)\|_{T,\Omega_0,\infty,p})\\ &\quad\le O(T)(\|w_2 - w_1\|_{{}_0\mathbb{E}_u(T,\Omega_0)} + \|\zeta_2 - \zeta_1\|_{{}_0\mathbb{E}_\tau(T,\Omega_0)}).\end{aligned}$$

In summary, we proved

$$\|G(w,\zeta)\|_{\mathbb{G}(T,\Omega_0)} \le O(T), \quad \|G(w,\zeta)\|_{T,\Omega_0,\infty,p} + \|\nabla G(w,\zeta)\|_{T,\Omega_0,p,p} \le C$$

and

$$\|G(w_2,\zeta_2) - G(w_1,\zeta_1)\|_{\mathbb{G}(T,\Omega_0)} \le O(T)(\|w_2 - w_1\|_{{}_0\mathbb{E}_u(T,\Omega_0)} + \|\zeta_2 - \zeta_1\|_{{}_0\mathbb{E}_\tau(T,\Omega_0)}).$$

Analysis of $H(w,\hat{\psi},\zeta)$

The nonlinearity H is the remaining component of N, which we have to analyse. The aim is to show the estimates

$$\|H(w,\hat{\psi},\zeta)\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} \leq CR^2 + O(T)$$

and

$$\begin{aligned}&\|H(w_2,\hat{\psi}_2,\zeta_2) - H(w_1,\hat{\psi}_1,\zeta_1)\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} \\ &\qquad\qquad \leq (CR + O(T))\|(w_2 - w_1, \hat{\psi}_2 - \hat{\psi}_1, \zeta_2 - \zeta_1)\|_{{}_0\mathbb{E}_u(T,\Omega_0)\times{}_0\mathbb{E}_{\hat{\pi}}(T,\Gamma_{F,0})\times{}_0\mathbb{E}_\tau(T,\Omega_0)}.\end{aligned}$$

To show the second estimate, we prove

$$\|DH(w,\hat{\psi},\zeta)\|_{\mathcal{L}({}_0\mathbb{E}_u(T,\Omega_0)\times{}_0\mathbb{E}_{\hat{\pi}}(T,\Gamma_{F,0})\times{}_0\mathbb{E}_\tau(T,\Omega_0),{}_0\mathbb{H}_u(T,\Gamma_{F,0}))} \leq CR + O(T)$$

and apply the mean value theorem.

We recall the definition (see (4.40))

$$H(w,\hat{\psi},\zeta) = H_w(w) + H_\psi(w,\hat{\psi}) + H_\zeta(w,\zeta).$$

and analyse the terms H_w, H_ψ, and H_ζ in the following separately.
$\mathbf{H_w}$: The term H_w decomposes in three summands (see (4.41))

$$H_w(w) = H_w^{(1)}(w) + H_w^{(2)}(w) + H_w^{(3)}(w),$$

with

$$\begin{aligned}H_w^{(1)}(w) &= -2(\alpha(|\mathcal{E}(v)|^2) - \alpha(|Ev_*|^2))Ew\nu_0,\\ H_w^{(2)}(w) &= -2\big(\alpha(|\mathcal{E}(v)|^2) - \alpha(|Ev_*|^2) - 2\alpha'(|Ev_*|^2)(Ev_* : Ew)\big)Ev_*\nu_0,\\ H_w^{(3)}(w) &= -\alpha(|\mathcal{E}(v)|^2)(B(I(v))^T\nabla v + (\nabla v)^T B(I(v)) + \nabla v B(I(v))^T + (\nabla v)^T B(I(v))^T\\ &\qquad + B(I(v))^T\nabla v B(I(v))^T + (\nabla v)^T B(I(v))B(I(v))^T)\nu_0.\end{aligned}$$

Further, we recall the definition (see (4.12))

$$\mathcal{E}(v) = Ev + \frac{1}{2}\big(B(I(v))^T\nabla v + (\nabla v)^T B(I(v))\big).$$

First, it should be noted that

$$\|v\|_{T,\Gamma_{F,0},\infty,\infty} + \|v\|_{\mathbb{H}_u(T,\Gamma_{F,0})} \leq C, \tag{4.75}$$

due to (4.51). By the proposition on pointwise multiplications (Proposition 1.16), the proposition on trace and extension theorems (Proposition 1.15), Lemma 4.4, and (4.51), it follows that

$$\mathcal{E}(v) \in \mathbb{H}_u(T,\Gamma_{F,0}) \quad \text{and} \quad \|\mathcal{E}(v)\|_{\mathbb{H}_u(T,\Gamma_{F,0})} + \|\mathcal{E}(v)\|_{T,\Gamma_{F,0},\infty,\infty} \leq C. \tag{4.76}$$

By the same arguments, taking additionally into account the proposition on Nemytskij operators and (4.76), we conclude that

$$H_w(w) \in {}_0\mathbb{H}_u(T,\Gamma_{F,0}).$$

The expression $\||\mathcal{E}(v)|^2 - |Ev_*|^2\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})}$ will play an important role in the analysis of H_w. The proposition on pointwise multiplications, the proposition on trace and extension theorems, Lemma 4.4, (4.75), and (4.76) imply

$$
\begin{aligned}
&\||\mathcal{E}(v)|^2 - |Ev_*|^2\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} \\
&\quad \le \|(\mathcal{E}(v) - Ev_* : \mathcal{E}(v) + Ev_*)\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} \\
&\quad \le C\|\mathcal{E}(v) - Ev_*\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})}(\|\mathcal{E}(v) + Ev_*\|_{\mathbb{H}_u(T,\Gamma_{F,0})} + \|\mathcal{E}(v) + Ev_*\|_{T,\Gamma_{F,0},\infty,\infty}) \\
&\quad \le C(\|Ew\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} + \|B(I(v))\nabla v\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})}) \\
&\quad \le C(\|w\|_{{}_0\mathbb{E}_u(T,\Omega_0)} + \|B(I(v))\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})}(\|v\|_{\mathbb{H}_u(T,\Gamma_{F,0})} + \|v\|_{T,\Gamma_{F,0},\infty,\infty})) \\
&\quad \le CR + O(T).
\end{aligned} \tag{4.77}
$$

We analyse each summand of H_w separately. Using the proposition on pointwise multiplications, the proposition on trace and extension theorems, Proposition 1.19, and (4.77), it follows that

$$
\begin{aligned}
&\|H_w^{(1)}(w)\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} \\
&\quad \le C\|\alpha(|\mathcal{E}(v)|^2) - \alpha(|Ev_*|^2)\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})}\|Ew\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} \\
&\quad \le C\||\mathcal{E}(v)|^2 - |Ev_*|^2\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})}\|Ew\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} \\
&\quad \le (CR + O(T))\|w\|_{{}_0\mathbb{E}_u(T,\Omega_0)} \\
&\quad \le CR^2 + O(T).
\end{aligned}
$$

Next, we analyse the Fréchet derivative of $H_w^{(1)}$. First, we examine the Fréchet derivative of $\mathcal{E}(v)$. By the product rule, the proposition on trace and extension theorems, the proposition on pointwise multiplications, Lemma 4.4, and (4.51), we deduce that

$$
\begin{aligned}
&\|D_w\mathcal{E}(v)[\bar{w}]\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} \\
&\quad \le \|E\bar{w}\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} + C\big(\|D_wB(I(v))[\bar{w}]\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})}(\|\nabla v\|_{\mathbb{H}_u(T,\Gamma_{F,0})} + \|\nabla v\|_{T,\Gamma_{F,0},\infty,\infty}) \\
&\qquad + \|B(I(v))\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})}\|\nabla\bar{w}\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})}\big) \\
&\quad \le C\|\bar{w}\|_{{}_0\mathbb{E}_u(T,\Omega_0)}, \quad \bar{w} \in {}_0\mathbb{E}_u(T,\Omega_0).
\end{aligned} \tag{4.78}
$$

By similar arguments and additionally taking into account the proposition on Nemytskij operators and (4.78), we infer

$$
\begin{aligned}
&\|DH_w^{(1)}(w)\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} \\
&\quad \le C(\|\alpha'(|\mathcal{E}(v)|^2)(\mathcal{E}(v) : D_w\mathcal{E}(v)[\bar{w}])Ew\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} + \|(\alpha(|\mathcal{E}(v)|^2) - \alpha(|Ev_*|^2))E\bar{w}\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})}) \\
&\quad \le C(\|\alpha'(|\mathcal{E}(v)|^2)\|_{\mathbb{H}_u^\infty(T,\Gamma_{F,0})}\|\mathcal{E}(v)\|_{\mathbb{H}_u^\infty(T,\Gamma_{F,0})}\|D_w\mathcal{E}(v)[\bar{w}]\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})}\|Ew\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} \\
&\qquad + \|(\alpha(|\mathcal{E}(v)|^2) - \alpha(|Ev_*|^2))\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})}\|E\bar{w}\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})}) \\
&\quad \le (CR + O(T))\|\bar{w}\|_{{}_0\mathbb{E}_u(T,\Omega_0)}, \quad \bar{w} \in {}_0\mathbb{E}_u(T,\Omega_0).
\end{aligned}
$$

To estimate $H_w^{(2)}(w)$, we will apply Proposition 1.19. We decompose $H_w^{(2)}(w)$ into a part $H_w^{(2,1)}(w)$, which fits in the setting of Proposition 1.19 and into a remainder part $H_w^{(2,2)}(w)$, more precisely we decompose $H_w^{(2)}(w)$ in the following way.

$$
\begin{aligned}
H_w^{(2)}(w) &= -2\big(\alpha(|\mathcal{E}(v)|^2) - \alpha(|Ev_*|^2) - 2\alpha'(|Ev_*|^2)(Ev_* : Ew)\big)Ev_*\nu_0 \\
&= H_w^{(2,1)}(w) + H_w^{(2,2)}(w),
\end{aligned}
$$

with

$$\begin{aligned}
H_w^{(2,1)}(w) &:= -2\big(\alpha(|\mathcal{E}(v)|^2) - \alpha(|Ev_*|^2) - \alpha'(|Ev_*|^2)(|\mathcal{E}(v)|^2 - |Ev_*|^2)\big)Ev_*\nu_0,\\
H_w^{(2,2)}(w) &:= -2\alpha'(|Ev_*|^2)\big(|\mathcal{E}(v)|^2 - |Ev_*|^2 - 2(Ev_* : Ew)\big)Ev_*\nu_0\\
&= -2\alpha'(|Ev_*|^2)\big(|Ew|^2 + |\mathcal{E}(v)|^2 - |Ev|^2\big)Ev_*\nu_0\\
&= -2\alpha'(|Ev_*|^2)\big(|Ew|^2 + (\mathcal{E}(v) - Ev : \mathcal{E}(v) - Ev) + 2(\mathcal{E}(v) - Ev : Ev)\big)Ev_*\nu_0\\
&= -2\alpha'(|Ev_*|^2)\big(|Ew|^2 + (Ev : B(I(v))^T\nabla v + (\nabla v)^T B(I(v)))\\
&\qquad + \frac{1}{4}|B(I(v))^T\nabla v + (\nabla v)^T B(I(v))|^2\big)Ev_*\nu_0.
\end{aligned}$$

Applying Proposition 1.19 and estimate (4.77) of $|\mathcal{E}(v)|^2 - |Ev_*|^2$, we can assert that

$$\|H_w^{(2,1)}(w)\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} \le C\||\mathcal{E}(v)|^2 - |Ev_*|^2\|^2_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} \le CR^2 + O(T).$$

Let us compute the Fréchet derivative $H_w^{(2,1)}$. By the proposition on Nemytskij operators, we see that

$$\begin{aligned}
DH_w^{(2,1)}(w)[\bar{w}] &= -4\big(\alpha'(|\mathcal{E}(v)|^2)(\mathcal{E}(v) : D_w\mathcal{E}(v)[\bar{w}]) - \alpha'(|Ev_*|^2)(\mathcal{E}(v) : D_w\mathcal{E}(v)[\bar{w}])\big)Ev_*\nu_0.\\
&= -4\big(\alpha'(|\mathcal{E}(v)|^2) - \alpha'(|Ev_*|^2)\big)(\mathcal{E}(v) : D_w\mathcal{E}(v)[\bar{w}])Ev_*\nu_0.
\end{aligned}$$

Thus, by the proposition on pointwise multiplications, Proposition 1.19, (4.76), (4.77), and estimate (4.78) for the Fréchet derivative $D_w\mathcal{E}(v)$, we conclude that

$$\begin{aligned}
&\|DH_w^{(2,1)}(w)[\bar{w}]\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})}\\
&\quad \le C\|(\alpha'(|\mathcal{E}(v)|^2) - \alpha'(|Ev_*|^2))\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})}\|(\mathcal{E}(v) : D_w\mathcal{E}(v)[\bar{w}])\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})}\\
&\quad \le C\||\mathcal{E}(v)|^2 - |Ev_*|^2\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})}\|\bar{w}\|_{{}_0\mathbb{E}_u(T,\Omega_0)}\\
&\quad \le (CR + O(T))\|\bar{w}\|_{{}_0\mathbb{E}_u(T,\Omega_0)}.
\end{aligned}$$

Thus, we proved

$$\|H_w^{(2,1)}(w)\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} \le CR^2 + O(T) \quad \text{and} \quad \|DH_w^{(2,1)}(w)\|_{\mathcal{L}({}_0\mathbb{E}_u(T,\Omega_0),{}_0\mathbb{H}_u(T,\Gamma_{F,0}))} \le CR + O(T).$$

We continue with the investigation of $H_w^{(2,2)}$. Each component of $H_w^{(2,2)}$ is a sum, where each summand is of one of the forms

$$\begin{aligned}
H_w^{(2,2,1)}(w) &= \alpha'(|Ev_*|^2)\partial_{j_1}w_{j_2}\partial_{k_1}w_{k_2}(Ev_*)_{l_1,l_2}\nu_{0,m_1},\\
H_w^{(2,2,2)}(w) &= \alpha'(|Ev_*|^2)\partial_{j_1}v_{j_2}\partial_{k_1}v_{k_1}b(I(v))(Ev_*)_{l_1,l_2}\nu_{0,m_1},
\end{aligned}$$

where $b\colon \mathbb{R}^{n\times n} \to \mathbb{R}$ is a smooth function with $b(0) = 0$ and $j_\lambda, k_\lambda, l_\lambda, m_\lambda \in \{1, \dots, n\}$, $\lambda = 1, \dots, 3$, are indices. By the proposition on trace and extension theorems, the proposition on pointwise multiplications, the proposition on Nemytskij operators, and (4.75), we conclude that

$$\begin{aligned}
\|H_w^{(2,2,1)}(w)\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} &\le \|\alpha'(|Ev_*|^2)\partial_{j_1}w_{j_2}\partial_{k_1}w_{k_2}\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})}\\
&\le C\|\nabla w\|^2_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})}\\
&\le C\|w\|^2_{{}_0\mathbb{E}_u(T,\Omega_0)} \le CR^2
\end{aligned}$$

and

$$\begin{aligned}
\|DH_w^{(2,2,1)}(w)[\bar{w}]\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} &\le \|D\alpha'(|Ev_*|^2)\partial_{j_1}w_{j_2}\partial_{k_1}w_{k_2}[\bar{w}]\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} \\
&\le C\|\nabla w\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})}\|\nabla \bar{w}\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} \\
&\le C\|\nabla w\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})}\|\nabla \bar{w}\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} \\
&\le CR\|\bar{w}\|_{{}_0\mathbb{E}_u(T,\Omega_0)}, \quad \bar{w} \in {}_0\mathbb{E}_u(T,\Omega_0).
\end{aligned}$$

By similar arguments and additionally taking into account Lemma 4.4, we have

$$\|H_w^{(2,2,2)}(w)\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} \le C(\|\nabla v\|_{{}_0\mathbb{H}(T,\Gamma_{F,0})} + \|\nabla v\|_{T,\Gamma_{F,0},\infty,\infty})^2\|b(I(v))\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} \le O(T).$$

Next, we analyse the Fréchet derivative of $H_w^{(2,2,2)}$. We compute

$$\begin{aligned}
&DH_w^{(2,2,2)}(w)[\bar{w}] \\
&\quad= \alpha'(|Ev_*|^2)\partial_{j_1}\bar{w}_{j_2}\partial_{k_1}v_{k_1}b(I(v))(Ev_*)_{l_1,l_2} + \alpha'(|Ev_*|^2)\partial_{j_1}v_{j_2}\partial_{k_1}\bar{w}_{k_1}b(I(v))(Ev_*)_{l_1,l_2} \\
&\qquad+ \alpha'(|Ev_*|^2)\partial_{j_1}v_{j_2}\partial_{k_1}v_{k_1}D_w b(I(v))[\bar{w}](Ev_*)_{l_1,l_2}, \quad \bar{w} \in {}_0\mathbb{E}_u(T,\Omega_0).
\end{aligned}$$

Application of the proposition on trace and extension theorems, the proposition on pointwise multiplications, Lemma 4.4, and (4.51) yields

$$\begin{aligned}
&\|DH_w^{(2,2,2)}(w)[\bar{w}]\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} \\
&\quad\le C\big((\|\nabla v\|_{\mathbb{H}_u(T,\Gamma_{F,0})} + \|\nabla v\|_{T,\Gamma_{F,0},\infty,\infty})\|b(I(v))\|_{{}_0\mathbb{H}(T,\Gamma_{F,0})}\|\nabla\bar{w}\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} \\
&\qquad+ (\|\nabla v\|_{\mathbb{H}_u(T,\Gamma_{F,0})} + \|\nabla v\|_{T,\Gamma_{F,0},\infty,\infty})^2\|D_w b(I(v))[\bar{w}]\|_{{}_0\mathbb{H}(T,\Gamma_{F,0})}\big) \\
&\quad\le O(T)\|\bar{w}\|_{{}_0\mathbb{E}_u(T,\Omega_0)}, \quad \bar{w} \in {}_0\mathbb{E}_u(T,\Omega_0).
\end{aligned}$$

In summary, we proved

$$\|H_w^{(2,2)}(w)\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} \le O(T) \quad \text{and} \quad \|DH_w^{(2,2)}(w)\|_{\mathcal{L}({}_0\mathbb{E}_u(T,\Omega_0),{}_0\mathbb{H}_u(T,\Gamma_{F,0}))} \le O(T).$$

Our next subject is $H_w^{(3)}$. Each component of $H_w^{(3)}$ is a sum, where each summand is of the form

$$H_w^{(3,1)}(w) = \alpha(|\mathcal{E}(v)|)\partial_{j_1}v_{j_2}b(I(v))\nu_{0,k},$$

where $b\colon \mathbb{R}^{n\times n} \to \mathbb{R}$ is a smooth function with $b(0) = 0$ and $j_1, j_2, k \in \{1,\dots,n\}$ are indices. By the proposition on pointwise multiplications, the proposition on Nemytskij operators, Lemma 4.4, (4.51), and (4.76), it follows that

$$\|H_w^{(3,1)}(w)\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} \le C(\|\nabla v\|_{\mathbb{H}_u(T,\Gamma_{F,0})} + \|\nabla v\|_{T,\Gamma_{F,0},\infty,\infty})\|b(I(v))\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} \le O(T).$$

We compute the Fréchet derivative of $H_w^{(3,1)}$. We have

$$\begin{aligned}
DH_w^{(3,1)}(w)[\bar{w}] &= \alpha'(|\mathcal{E}(v)|^2)(\mathcal{E}(v) : D_w\mathcal{E}(v)[\bar{w}])(\partial_{j_1}v_{j_2})b(I(v))\nu_{0,k} + \alpha(|\mathcal{E}(v)|)\partial_{j_1}\bar{w}_{j_2}b(I(v))\nu_{0,k} \\
&\quad+ \alpha(|\mathcal{E}(v)|)(\partial_{j_1}v_{j_2})D_w b(I(v))[\bar{w}]\nu_{0,k}, \quad \bar{w} \in {}_0\mathbb{E}_u(T,\Omega_0).
\end{aligned}$$

By similar arguments as above, and additionally taking into account Lemma 4.4, we deduce that

$$\begin{aligned}
&\|DH_w^{(3,1)}(w)[\bar{w}]\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} \\
&\quad\le C(\|D_w\mathcal{E}(v)[\bar{w}]\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})}\|b(I(v))\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} + \|\nabla\bar{w}\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})}\|b(I(v))\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} \\
&\qquad + \|D_w b(I(v))[\bar{w}]\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})}) \\
&\quad\le O(T)\|\bar{w}\|_{{}_0\mathbb{E}_u(T,\Omega_0)}, \quad \bar{w}\in {}_0\mathbb{E}_u(T,\Omega_0).
\end{aligned}$$

Hence,

$$\|H_w^{(3)}(w)\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} \le O(T) \quad \text{and} \quad \|DH_w^{(3)}(w)\|_{\mathcal{L}({}_0\mathbb{E}_u(T,\Omega_0),{}_0\mathbb{H}_u(T,\Gamma_{F,0}))} \le O(T).$$

In summary, we proved

$$\|H_w(w)\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} \le CR^2 + O(T) \quad \text{and} \quad \|DH_w(w)\|_{\mathcal{L}({}_0\mathbb{E}_u(T,\Omega_0),{}_0\mathbb{H}_u(T,\Gamma_{F,0}))} \le CR + O(T).$$

$\mathbf{H}_\psi$: The nonlinearity H_ψ is the next subject. By the proposition on embedding theorems, we have

$$\|\hat{\psi}\|_{T,\Gamma_{F,0},\infty,\infty} + \|\hat{\psi}\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} + \|\theta_*\|_{T,\Gamma_{F,0},\infty,\infty} + \|\theta_*\|_{\mathbb{H}_u(T,\Gamma_{F,0})} \le C.$$

We recall the definition (see (4.42))

$$H_\psi(w,\hat{\psi}) = -B(I(w+v_*))^T(\theta_* + \hat{\psi})\nu_0.$$

The proposition on pointwise multiplications and Lemma 4.4 imply

$$\begin{aligned}
\|H_\psi(w,\hat{\psi})\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} &\le \|B(I(w+v_*))\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})}(\|\theta_*+\hat{\psi}\|_{\mathbb{H}_u(T,\Gamma_{F,0})} + \|\theta_*+\hat{\psi}\|_{T,\Gamma_{F,0},\infty,\infty}) \\
&\le O(T),
\end{aligned}$$

as well as

$$\begin{aligned}
&\|DH_\psi(w,\hat{\psi})[\bar{w},\bar{\psi}]\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} \\
&\quad\le C(\|D_w B(I(w+v_*))[\bar{w}]\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})}(\|\theta_*+\hat{\psi}\|_{\mathbb{H}_u(T,\Gamma_{F,0})} + \|\theta_*+\hat{\psi}\|_{T,\Gamma_{F,0},\infty,\infty}) \\
&\qquad + \|B(I(w+v_*))\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})}\|\bar{\psi}\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})}) \\
&\quad\le O(T)(\|\bar{w}\|_{{}_0\mathbb{E}_u(T,\Omega_0)} + \|\bar{\psi}\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})}), \quad (\bar{w},\bar{\psi})\in {}_0\mathbb{E}_u(T,\Omega_0)\times {}_0\mathbb{E}_{\hat{\psi}}(T,\Gamma_{F,0}),
\end{aligned}$$

and hence

$$\|DH_\psi(w,\hat{\psi})\|_{\mathcal{L}({}_0\mathbb{E}_u(T,\Omega_0)\times {}_0\mathbb{E}_{\hat{\psi}}(T,\Gamma_{F,0}),{}_0\mathbb{H}_u(T,\Gamma_{F,0}))} \le O(T).$$

$\mathbf{H}_\zeta$: Let us examine H_ζ. We recall the definition

$$H_\zeta(w,\zeta) = -(\mu(\eta)-\mu(\tau_0))\nu_0 - \mu(\eta)B(I(v))^T\nu_0.$$

We have

$$\|\zeta\|_{T,\Gamma_{F,0},\infty,\infty} + \|\zeta\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} + \|\tau_0\|_{\Gamma_{F,0},\infty} + \|\tau_0\|_{\mathbb{H}_u(T,\Gamma_{F,0})} \le C.$$

By the proposition on pointwise multiplications, the proposition on Nemytskij operators and Proposition 1.19, it follows that

$$\begin{aligned}\|H_\zeta(w,\zeta)\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} &\le C(\|\mu(\eta)-\mu(\tau_0)\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} + \|\mu(\eta)B(I(v))^T\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})})\\ &\le C(\|\eta-\tau_0\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} + \|\mu(\eta)\|_{\mathbb{H}_u^\infty(T,\Gamma_{F,0})}\|B(I(v))\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})})\\ &\le C(\|\eta-\tau_0\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} + C\|B(I(v))\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})}).\end{aligned}$$

Since the trace operator is continuous and the operator norm can be estimated uniformly in T, $0<T<T_0$ (see Proposition 1.15), we have

$$\begin{aligned}\|H_\zeta(w,\zeta)\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} &\le C\|\eta-\tau_0\|_{{}_0H_p^{\frac{1}{2}}(0,T;L_p(\Omega_0))\cap L_p(0,T;H_p^1(\Omega_0))} + O(T)\\ &\le O(T)(\|\eta-\tau_0\|_{{}_0H_r^1(0,T;L_p(\Omega_0))\cap L_\infty(0,T;H_p^1(\Omega_0))}+1)\\ &\le O(T),\end{aligned}$$

due to Lemma 4.4. The Fréchet derivative of H_ζ can be estimated by similar arguments

$$\begin{aligned}&\|DH_\zeta(w,\zeta)[(\bar w,\bar\zeta)]\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})}\\ &\quad\le C(\|D_\zeta\mu(\eta)[\bar\zeta]\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} + \|D_\zeta\mu(\eta)[\bar\zeta]\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})}\|B(I(v))\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})}\\ &\qquad + \|\mu(\zeta)\|_{\mathbb{H}_u^\infty(T,\Gamma_{F,0})}\|D_wB(I(v))[\bar w]\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})})\\ &\quad\le (1+O(T))\|\mu'(\zeta)\|_{\mathbb{H}_u^\infty(T,\Gamma_{F,0})}\|\bar\zeta\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} + O(T)\|\mu(\zeta)\|_{\mathbb{H}_u^\infty(T,\Gamma_{F,0})}\|\bar w\|_{{}_0\mathbb{E}_u(T,\Omega_0)}\\ &\quad\le O(T)(\|\bar w\|_{{}_0\mathbb{E}_u(T,\Omega_0)} + \|\bar\zeta\|_{{}_0\mathbb{E}_\zeta(T,\Omega_0)}), \quad (\bar w,\bar\zeta)\in {}_0\mathbb{E}_u(T,\Omega_0)\times{}_0\mathbb{E}_\zeta(T,\Omega_0),\end{aligned}$$

where we once more used the proposition on trace and embedding theorems, the proposition on pointwise multiplications, the proposition on Nemytskij operators, and Lemma 4.4. Thus, we proved

$$\|DH_\zeta(w,\zeta)\|_{\mathcal{L}({}_0\mathbb{E}_u(T,\Omega_0)\times{}_0\mathbb{E}_\zeta(T,\Omega_0),{}_0\mathbb{H}(T,\Gamma_{F,0}))} \le O(T).$$

In summary, we obtain

$$\|H(w,\hat\psi,\zeta)\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})} \le CR^2 + O(T),$$

and

$$\|DH(w,\hat\psi,\zeta)\|_{\mathcal{L}({}_0\mathbb{E}_u(T,\Omega_0)\times{}_0\mathbb{E}_{\hat\pi}(T,\Gamma_{F,0})\times{}_0\mathbb{E}_\tau(T,\Omega_0),{}_0\mathbb{H}_u(T,\Gamma_{F,0}))} \le CR + O(T).$$

Applying the mean value theorem yields

$$\begin{aligned}&\|H(w_2,\hat\psi_2,\zeta_2) - H(w_1,\hat\psi_1,\zeta_1)\|_{{}_0\mathbb{H}_u(T,\Gamma_{F,0})}\\ &\qquad\le (CR+O(T))\|(w_2-w_1,\hat\psi_2-\hat\psi_1,\zeta_2-\zeta_1)\|_{{}_0\mathbb{E}_u(T,\Omega_0)\times{}_0\mathbb{E}_{\hat\pi}(T,\Gamma_{F,0})\times{}_0\mathbb{E}_\tau(T,\Omega_0)}.\end{aligned}$$

This completes the proof. □

Application of the fixed point argument and completion of the proof

We are now in a position to apply the standard version of the contraction mapping principle. Let $R_0 = T_0 = 1$, $0 < R < R_0$, $0 < T < T_0$, and $(w, \psi, \hat{\psi}, \zeta), (w_j, \psi_j, \hat{\psi}_j, \zeta_j) \in \overline{B}_{{}_0\mathbb{E}(T)}(0, R)$, $j = 1, 2$. By the solvability result on the generalized Stokes equation (Proposition 1.8), estimate (4.45) for ζ, and Lemma 4.3, we obtain

$$\begin{aligned}
\|\Phi(w, \psi, \hat{\psi}, \zeta)\|_{{}_0\mathbb{E}(T)} &= \|\tilde{\Phi}_0(n_* + N(w, \psi, \hat{\psi}, \zeta))\|_{{}_0\mathbb{E}(T)} \\
&\le C(\|n_*\|_{{}_0\mathbb{F}(T)} + \|N(w, \psi, \hat{\psi}, \zeta)\|_{{}_0\mathbb{F}(T)}) \\
&\le CR^2 + O(T),
\end{aligned}$$

and

$$\begin{aligned}
&\|\Phi(w_2, \psi_2, \hat{\psi}_2, \zeta_2) - \Phi(w_1, \psi_1, \hat{\psi}_1, \zeta_1)\|_{{}_0\mathbb{E}(T)} \\
&\quad \le \|\tilde{\Phi}_0(N(w_2, \psi_2, \hat{\psi}_2, \zeta_2) - N(w_1, \psi_1, \hat{\psi}_1, \zeta_1))\|_{{}_0\mathbb{E}(T)} \\
&\quad \le C\|N(w_2, \psi_2, \hat{\psi}_2, \zeta_2) - N(w_1, \psi_1, \hat{\psi}_1, \zeta_1)\|_{{}_0\mathbb{F}(T)} \\
&\quad \le (CR + O(T))\|(w_2 - w_1, \psi_2 - \psi_2, \hat{\psi}_2 - \hat{\psi}_1, \zeta_2 - \zeta_1)\|_{{}_0\mathbb{E}(T)},
\end{aligned}$$

where $O\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a function which is independent of R with $O(t) \to 0$ for $t \to 0$. Choosing first $R > 0$ and then $T > 0$ sufficiently small, it follows that

$$\|\Phi(w, \psi, \hat{\psi}, \zeta)\|_{{}_0\mathbb{E}(T)} \le R,$$

and

$$\|\Phi(w_2, \psi_2, \hat{\psi}_2, \zeta_2) - \Phi(w_2, \psi_2, \hat{\psi}_2, \zeta_2)\|_{{}_0\mathbb{E}(T)} \le \frac{1}{2}\|(w_2 - w_1, \psi_2 - \psi_2, \hat{\psi}_2 - \hat{\psi}_1, \zeta_2 - \zeta_1)\|_{{}_0\mathbb{E}(T)}.$$

Since $0 \in B_{{}_0\mathbb{E}(T)}(0, R)$, it follows that $B_{{}_0\mathbb{E}(T)}(0, R) \neq \emptyset$. Applying the contraction mapping principle yields a unique fixed point $(w, \psi, \hat{\psi}, \zeta) \in {}_0\mathbb{E}(T)$, or equivalently a unique solution of (4.44). It remains to show the additional regularity of the elastic part of the stress. In the previous lemma we proved $G(w, \zeta) \in L_\infty(0, T; L_p(\Omega_0)) \cap L_p(0, T; H^1_p(\Omega_0))$. Since $\partial_t \zeta = G(w, \zeta)$, it follows that

$$\zeta \in {}_0W^1_\infty(0, T; L_p(\Omega_0)) \cap {}_0H^1_p(0, T; H^1_p(\Omega)).$$

$\square$

Appendix A

Transport equation

A proof of the proposition on the transport equation (Proposition 1.10) is given. Firstly, we sketch the main ideas.

Sketch of the proof

The first step is to prove the a-priori estimate stated in Proposition 1.10 **(b)**. This also proves the uniqueness of the solution. In the next step, we construct the solution in the case $\Omega = \mathbb{R}^n$. We approximate the data by smooth functions and apply the method of characteristic curves to solve the regularized problem (for a thorough treatment of the method of characteristic curves, we refer the reader to Evans [Eva10, Section 3.2]). In the case of the whole space, it is now sufficient to show that the series of approximated solution converges and fulfills the original problem (1.12). Since we did not impose any boundary condition, we can solve the transport equation on a domain Ω by extending the data to the whole space, solving the equation on the whole space, and restricting the solution onto the domain Ω.

Proof of Proposition 1.10. Let $\tau \in L_\infty(0,T;H_q^1(\Omega)) \cap \widehat{W}_r^1(0,T;L_q(\Omega))$ be a solution of (1.12). We prove the estimate in assertion **(b)**, i.e.

$$\|\tau\|_{T,\Omega,\infty,q} \leq (\|\tau_0\|_{\Omega,q} + \|g\|_{T,\Omega,1,q}) e^{C_{\mathrm{Tra}}^{(2)} \|\operatorname{div} u\|_{L_1(0,T;H_q^1(\Omega))}}.$$

We fix $t \in (0,T)$, multiply equation (1.12) with $|\tau(t)|^{q-2}\tau(t)$, and integrate over Ω:

$$\text{(A.1)} \qquad \int_\Omega \big((\partial_t\tau)(t) : |\tau(t)|^{q-2}\tau(t)\big)dx + \int_\Omega \big(u(t).\nabla\tau(t) : |\tau(t)|^{q-2}\tau(t)\big)dx = \int_\Omega \big(g(t) : |\tau(t)|^{q-2}\tau(t)\big)dx.$$

We investigate each appearing summand separately. By the chain rule, we compute

$$\text{(A.2)} \qquad \frac{d}{dt}\|\tau(t)\|_{\Omega,q}^q = \int_\Omega \partial_t|\tau(t)|^q dx = q\int_\Omega |\tau(t)|^{q-1}\partial_t|\tau(t)|dx = q\int_\Omega |\tau(t)|^{q-2}(\tau(t) : \partial_t\tau(t))dx.$$

According to $u \cdot \nu = 0$ on $(0,T) \times \partial\Omega$ and $\partial_j |\tau(t)|^q = q|\tau(t)|^{q-2}(\tau(t) : \partial_j \tau(t))$, the second term on the left-hand side of (A.1) can be simplified to

$$\begin{aligned}
\int_\Omega \big(u(t).\nabla\tau(t) : |\tau(t)|^{q-2}\tau(t)\big)dx &= \sum_{j=1}^n \int_\Omega u_j(t)|\tau(t)|^{q-2}(\tau(t) : \partial_j\tau(t))dx \\
&= \frac{1}{q}\sum_{j=1}^n \int_\Omega u_j(t)\partial_j|\tau(t)|^q dx \\
&= -\frac{1}{q}\int_\Omega |\tau(t)|^q \operatorname{div} u(t)dx + \frac{1}{q}\int_{\partial\Omega} u(t)\cdot\nu|\tau(t)|^q dx \\
&= -\frac{1}{q}\int_\Omega |\tau(t)|^q \operatorname{div} u(t)dx.
\end{aligned} \tag{A.3}$$

Furthermore, to estimate the right-hand side, we use the identity

$$\||\tau(t)|^{q-2}\tau(t)\|_{\Omega,q'} = \Big(\int_\Omega |\tau(t)|^{(q-1)q'}dx\Big)^{\frac{1}{q'}} = \|\tau(t)\|_{\Omega,q}^{\frac{q}{q'}} = \|\tau(t)\|_{\Omega,q}^{q-1},$$

where $1 < q' < \infty$ with $\frac{1}{q'} + \frac{1}{q} = 1$ (and thus $(q-1)q' = q$). Hence, applying Hölder's inequality and taking into account (A.1)–(A.3), we obtain

$$\begin{aligned}
\frac{1}{q}\frac{d}{dt}\|\tau(t)\|_{\Omega,q}^q &= \int_\Omega |\tau(t)|^{q-2}(\tau(t) : \partial_t\tau(t))dx \\
&= \int_\Omega \big(g(t) : |\tau(t)|^{q-2}\tau(t)\big)dx - \int_\Omega \big(u(t).\nabla\tau(t) : |\tau(t)|^{q-2}\tau(t)\big)dx \\
&\le \|g(t)\|_{\Omega,q}\|\tau(t)\|_{\Omega,q}^{q-1} + \frac{1}{q}\|\operatorname{div} u(t)\|_{\Omega,\infty}\|\tau(t)\|_{\Omega,q}^q.
\end{aligned}$$

By Sobolev's embedding theorem and the division of the result by $\|\tau(t)\|_q^{q-1}$, we deduce that

$$\frac{d}{dt}\|\tau(t)\|_{\Omega,q} \le \|g(t)\|_{\Omega,q} + C\|\operatorname{div} u(t)\|_{H_q^1(\Omega)}\|\tau(t)\|_{\Omega,q},$$

and, due to Gronwall's Lemma (Proposition 1.21) and Hölder's inequality, we infer

$$\|\tau\|_{T,\Omega,\infty,q} \le \big(\|\tau_0\|_{\Omega,q} + \|g\|_{T,\Omega,1,q}\big)e^{CT^{\frac{p-1}{p}}\|\operatorname{div} u\|_{L_p(0,T;H_q^1(\Omega))}}. \tag{A.4}$$

This proves the assertion **(b)** and the uniqueness of the solution (if it exists), since the equation is linear.

Next, we construct the solution in the case $\Omega = \mathbb{R}^n$. We extend u and g by zero for $t \notin [0,T]$. This extension is again denoted by u and g. Thus, we assume

$$u \in L_p(\mathbb{R}; H_q^2(\mathbb{R}^n)) \cap L_r(\mathbb{R}; L_\infty(\mathbb{R}^n)),\ g \in L_1(\mathbb{R}; H_q^1(\mathbb{R}^n)) \cap L_r(\mathbb{R}; L_q(\mathbb{R}^n)),\ \text{and } \tau_0 \in H_q^1(\mathbb{R}^n).$$

The aim here is to apply the method of characteristic curves. Since the coefficients are not regular enough to apply this method directly, we approximate them first by smooth function. Let

$$\varphi_\varepsilon^{(1)} \in C_c^\infty(\mathbb{R}\times\mathbb{R}^n) \quad \text{and} \quad \varphi_\varepsilon^{(2)} \in C_c^\infty(\mathbb{R}^n), \quad 0 < \varepsilon < 1,$$

be two mollifiers, and we define the regularized data

$$u_\varepsilon := u * \varphi_\varepsilon^{(1)}, \quad g_\varepsilon := g * \varphi_\varepsilon^{(1)}, \quad \text{and} \quad \tau_{0,\varepsilon} := \tau_0 * \varphi_\varepsilon^{(2)}, \quad 0 < \varepsilon < 1.$$

Fix $0 < \varepsilon < 1$. Next, we solve the system with the regularized data, i.e.

$$\left\{ \begin{array}{rcll} \partial_t \tau_\varepsilon + u_\varepsilon . \nabla \tau_\varepsilon &=& g_\varepsilon & \text{in } \mathbb{R}_+ \times \mathbb{R}^n, \\ \tau_\varepsilon(0) &=& \tau_{0,\varepsilon} & \text{in } \mathbb{R}^n, \end{array} \right. \tag{A.5}$$

using the standard method of characteristic curves. For $x \in \mathbb{R}^n$, we define the characteristic curve $\Psi_\varepsilon(\,\cdot\,; x)$, as the solution of the ordinary differential equation

$$\frac{d}{dt}\Psi_\varepsilon(t;x) = u_\varepsilon(t, \Psi_\varepsilon(t;x)), \quad t > 0, \qquad \Psi_\varepsilon(0;x) = x. \tag{A.6}$$

Integrating this equation, we obtain an equivalent representation of the characteristic curve

$$\Psi_\varepsilon(t;x) = x + \int_0^t u_\varepsilon(t', \Psi_\varepsilon(t';x))dt', \quad t > 0. \tag{A.7}$$

Next, we show that $\Psi_\varepsilon(t;\,\cdot\,)\colon \mathbb{R}^n \to \mathbb{R}^n$ is a C^2-diffeomorphism for $t > 0$. The first step is to show this property for a sufficiently small value of $t > 0$. The Jacobi matrix in the spatial components of the characteristic curve J_{Ψ_ε} solves the ordinary matrix differential equation

$$\frac{d}{dt}J_{\Psi_\varepsilon}(t;x) = (J_{u_\varepsilon})(t; \Psi_\varepsilon(t;x))J_{\Psi_\varepsilon}(t;x), \quad t > 0, \qquad J_{\Psi_\varepsilon}(0;x) = 1, \quad x \in \mathbb{R}^n,$$

where J_{u_ε} is the Jacobi matrix in the spatial components of u_ε. Integrating this equation, we obtain the equivalent integral equation

$$J_{\Psi_\varepsilon}(t;x) = 1 + \int_0^t (J_{u_\varepsilon})(t'; \Psi_\varepsilon(t';x))J_{\Psi_\varepsilon}(t';x)dt', \quad t > 0, \quad x \in \mathbb{R}^n.$$

Hence, we deduce that

$$\begin{aligned} \sup_{0<t'<t} \|J_{\Psi_\varepsilon}(t';\,\cdot\,)\|_{\mathbb{R}^n,\infty} &\le 1 + \int_0^t \|J_{u_\varepsilon}(s)\|_{\mathbb{R}^n,\infty} \|J_{\Psi_\varepsilon}(s;\,\cdot\,)\|_{\mathbb{R}^n,\infty} ds \\ &\le 1 + Ct^{\frac{p-1}{p}} \|u_\varepsilon\|_{L_p(\mathbb{R};H_q^2(\mathbb{R}^n))} \sup_{0<t'<t} \|J_{\Psi_\varepsilon}(t';\,\cdot\,)\|_{\mathbb{R}^n,\infty}, \quad t > 0, \end{aligned}$$

by Sobolev's embedding theorem and Hölder's inequality. We can choose $\eta > 0$, such that

$$\sup_{0<t<\eta} \|J_{\Psi_\varepsilon}(t;\,\cdot\,)\|_{\mathbb{R}^n,\infty} \le \frac{1}{2}, \tag{A.8}$$

since $\|u_\varepsilon\|_{L_p(\mathbb{R};H_q^2(\mathbb{R}^n))}$ is bounded. On the same way, it follows that

$$\|1 - J_{\Psi_\varepsilon}(t;\,\cdot\,)\|_{\mathbb{R}^n,\infty} = \int_0^t \|J_{u_\varepsilon}(s)\|_{\mathbb{R}^n,\infty} \|J_{\Psi_\varepsilon}(s;\,\cdot\,)\|_{\mathbb{R}^n,\infty} ds \le Ct^{\frac{p-1}{p}} \|u_\varepsilon\|_{L_p(\mathbb{R};H_q^2(\mathbb{R}^n))}, \quad 0 < t < \eta.$$

Hence, there exists a constant $0 < \eta_1 \le \eta$, such that the matrix $J_{\Psi_\varepsilon}(t,x)$ is invertible for $x \in \mathbb{R}^n$ and $0 < t < \eta_1$ due to the Neumann series. It should be noted, that η and η_1 are independent of Ψ_ε. By the theorem on local solvability, we deduce that $\Psi_\varepsilon(t;\,\cdot\,)\colon \mathbb{R}^n \to \mathbb{R}^n$ is locally a C^2-diffeomorphism. Next, we apply the contraction mapping principle to prove, that $\Psi_\varepsilon(t;\,\cdot\,)\colon \mathbb{R}^n \to \mathbb{R}^n$ is a bijection, provided that $t > 0$ is sufficiently small. For a given $y \in \mathbb{R}^n$, we have to find a unique solution $x \in \mathbb{R}^n$ of the equation $\Psi_\varepsilon(t,x) = y$. This is equivalent to the fixed point problem (see (A.7))

$$\Phi_{\varepsilon,t,y}(x) := y - \int_0^t u_\varepsilon(t', \Psi_\varepsilon(t';x))dt' = x, \quad x \in \mathbb{R}^n.$$

Since $\Psi_\varepsilon(t,\,\cdot\,)$ depends continuously on the initial data $x \in \mathbb{R}^n$, we deduce that

$$\begin{aligned} |\Phi_{\varepsilon,t,y}(x_2) - \Phi_{\varepsilon,t,y}(x_1)| &\le \int_0^t |u_\varepsilon(t', \Psi_\varepsilon(t';x_2)) - u_\varepsilon(t', \Psi_\varepsilon(t';x_1))|dt' \\ &\le \int_0^t \|\nabla u_\varepsilon(t')\|_{\mathbb{R}^n,\infty} \|\nabla \Psi_\varepsilon(t';\,\cdot\,)\|_{\mathbb{R}^n,\infty} ds |x_2 - x_1| \\ &\le t^{\frac{p-1}{p}} \|u_\varepsilon\|_{L_p(\mathbb{R};H^2_q(\mathbb{R}^n))} \sup_{0<t'<t} \|\nabla \Psi_\varepsilon(t';\,\cdot\,)\|_{\mathbb{R}^n,\infty} |x_2 - x_1| \\ &\le t^{\frac{p-1}{p}} \|u_\varepsilon\|_{L_p(\mathbb{R};H^2_q(\mathbb{R}^n))} |x_2 - x_1| \quad x_1, x_2 \in \mathbb{R}^n, \quad 0 < t < \eta, \end{aligned}$$

by the mean value theorem and the boundedness of $\sup_{0<t'<\eta} \|\nabla \Psi_\varepsilon(t';\,\cdot\,)\|_{\mathbb{R}^n,\infty}$ (see (A.8)). We can choose $0 < \eta_2 < \eta$, such that $\Phi_{\varepsilon,t,y}$ is a contraction, provided that $0 < t < \eta_2$. By the contraction mapping principle, we obtain that $\Psi_\varepsilon(t,\,\cdot\,)\colon \mathbb{R}^n \to \mathbb{R}^n$ is bijective for $0 < t < \eta_2$. In summary, we proved the existence of $\bar{\eta} := \min\{\eta_1, \eta_2\}$, which does not depend on Ψ_ε, such that the map $\Psi_\varepsilon(t,\,\cdot\,)\colon \mathbb{R}^n \to \mathbb{R}^n$ is a C^2-diffeomorphism, provided that $0 < t < \bar{\eta}$. Since $\Psi_\varepsilon(\,\cdot\,;x)$ is the solution of the unique solvable ordinary differential equation (A.5), we have the semigroup property $\Psi_\varepsilon(t_1 + t_2; x) = \Psi_\varepsilon(t_1; \Psi_\varepsilon(t_2;x))$ for $t_1, t_2 > 0$ and $x \in \mathbb{R}^n$. By this property, it follows that $\Psi_\varepsilon(t;\,\cdot\,)$ is a C^2-diffeomorphism for all $t > 0$. We denote its inverse by $\Xi_\varepsilon(t;\,\cdot\,)$, $t > 0$.

Now, we construct the solution of equation (A.5). For $x \in \mathbb{R}^n$ let $z_\varepsilon(\,\cdot\,;x)$ be the unique solution of the equation

$$\text{(A.9)} \qquad \frac{d}{dt} z_\varepsilon(t;x) = g_\varepsilon(t, \Psi_\varepsilon(t;x)), \quad t > 0, \qquad z_\varepsilon(0;x) = \tau_{0,\varepsilon}(x).$$

We define $\tau_\varepsilon(t,x) := z_\varepsilon(t; \Xi_\varepsilon(t;x))$ and show that τ_ε solves (A.5) for $t > 0$. Differentiating the identity $z_\varepsilon(t;x) = \tau_\varepsilon(t, \Psi_\varepsilon(t;x))$ and taking into account (A.6) and (A.9), it follows that

$$g_\varepsilon(t, \Psi_\varepsilon(t;x)) = \frac{d}{dt} z_\varepsilon(t;x) = (\partial_t \tau)(t, \Psi_\varepsilon(t;x)) + u(t, \Psi_\varepsilon(t;x)).(\nabla\tau)(t, \Psi_\varepsilon(t;x)).$$

Furthermore, $\tau_\varepsilon(0,x) = z_\varepsilon(0;x) = \tau_{0,\varepsilon}(x)$, and hence τ_ε solves (A.5).

Last, we show that the approximated solution τ_ε actually converges in a suitable sense and that the limit is the solution of the original problem (1.12) in the case that $\Omega = \mathbb{R}^n$. Let from now on $T > 0$. We show the two a-priori estimates stated in Proposition 1.10 **(a)** for τ_ε, i.e.

$$\begin{aligned} \|\tau_\varepsilon\|_{L_\infty(0,T;H^1_q(\mathbb{R}^n))} &\le C^{(1)}_{\mathrm{Tra}} \big(\|\tau_{0,\varepsilon}\|_{H^1_q(\mathbb{R}^n)} + \|g_\varepsilon\|_{L_1(0,T;H^1_p(\mathbb{R}^n))} \big) e^{C^{(1)}_{\mathrm{Tra}} T^{\frac{p-1}{p}} \|u_\varepsilon\|_{L_p(0,T;H^2_q(\mathbb{R}^n))}}, \\ \|\partial_t \tau_\varepsilon\|_{T,\mathbb{R}^n,r,q} &\le \|g_\varepsilon\|_{T,\mathbb{R}^n,r,q} + \|u_\varepsilon\|_{T,\mathbb{R}^n,r,\infty} \|\tau_\varepsilon\|_{L_\infty(0,T;H^1_q(\mathbb{R}^n))}. \end{aligned}$$

By construction, it holds

$$\tau_\varepsilon \in C([0,T], C(\mathbb{R}^n)) \cap C((0,T); C^2(\mathbb{R}^n)) \cap C^1((0,T); C^1(\mathbb{R}^n)).$$

Fix $0 < t < T$. Differentiating (A.5) with respect to x_m, $m = 1, \dots, n$, and testing the result with $|\partial_m \tau_\varepsilon(t)|^{q-2}\partial_m \tau_\varepsilon(t)$ yields

(A.10)
$$\int_{\mathbb{R}^n} |\partial_m \tau_\varepsilon(t)|^{q-2}\big((\partial_t \partial_m \tau_\varepsilon)(t) : \partial_m \tau_\varepsilon(t)\big)dx + \int_{\mathbb{R}^n} |\partial_m \tau_\varepsilon(t)|^{q-2}\big((\partial_m u_\varepsilon)(t).\nabla \tau_\varepsilon(t) : \partial_m \tau_\varepsilon(t)\big)dx$$
$$+ \int_{\mathbb{R}^n} |\partial_m \tau_\varepsilon(t)|^{q-2}\big(u_\varepsilon(t).(\nabla \partial_m \tau_\varepsilon)(t) : \partial_m \tau_\varepsilon(t)\big)dx = \int_{\mathbb{R}^n} |\partial_m \tau_\varepsilon(t)|^{q-2}\big((\partial_m g_\varepsilon)(t) : \partial_m \tau_\varepsilon(t)\big)dx.$$

We investigate each term. Applying the chain rule, we obtain

$$\begin{aligned}
\frac{d}{dt}\|\partial_m \tau_\varepsilon(t)\|^q_{\mathbb{R}^n,q} &= \int_{\mathbb{R}^n} \partial_t |\partial_m \tau_\varepsilon(t)|^q dx \\
&= q\int_{\mathbb{R}^n} |\partial_m \tau_\varepsilon(t)|^{q-1}\partial_t|\partial_m \tau_\varepsilon(t)|dx \\
&= q\int_{\mathbb{R}^n} |\partial_m \tau_\varepsilon(t)|^{q-2}\big((\partial_m \tau_\varepsilon)(t) : (\partial_t \partial_m \tau_\varepsilon)(t)\big)dx.
\end{aligned}$$

By $\partial_j|\partial_m \tau_\varepsilon(t)|^q = q|\partial_m \tau_\varepsilon(t)|^{q-2}((\partial_m \tau_\varepsilon)(t) : \partial_j \partial_m \tau_\varepsilon(t))$, the third summand on the left-hand side of (A.10) simplifies to

$$\begin{aligned}
\int_{\mathbb{R}^n} |\partial_m \tau_\varepsilon(t)|^{q-2}\big(u_\varepsilon(t).(\nabla \partial_m \tau_\varepsilon)(t) : \partial_m \tau_\varepsilon(t)\big)dx &= \frac{1}{q}\sum_{j=1}^n \int_{\mathbb{R}^n} u_{\varepsilon,j}(t)\partial_j|\partial_m \tau_\varepsilon(t)|^q dx \\
&= -\frac{1}{q}\int_{\mathbb{R}^n} \operatorname{div} u_\varepsilon(t)|\partial_m \tau_\varepsilon(t)|^q dx.
\end{aligned}$$

Further, we need the identity

$$\||\partial_m \tau_\varepsilon(t)|^{q-2}\partial_m \tau_\varepsilon(t)\|_{\mathbb{R}^n,q'} = \Big(\int_{\mathbb{R}^n} |\partial_m \tau_\varepsilon(t)|^{(q-1)q'}dx\Big)^{1/q'} = \|\partial_m \tau_\varepsilon(t)\|^{\frac{q}{q'}}_{\mathbb{R}^n,q} = \|\partial_m \tau_\varepsilon(t)\|^{q-1}_{\mathbb{R}^n,q},$$

where $1 < q' < \infty$ with $\frac{1}{q'} + \frac{1}{q} = 1$. In summary, we obtain

$$\begin{aligned}
\frac{1}{q}\frac{d}{dt}\|\partial_m \tau_\varepsilon(t)\|^q_{\mathbb{R}^n,q} &= \int_{\mathbb{R}^n} |\partial_m \tau_\varepsilon(t)|^{q-2}\big((\partial_t \partial_m \tau_\varepsilon)(t) : \partial_m \tau_\varepsilon(t)\big)dx \\
&\leq C\int_{\mathbb{R}^n} |\nabla \tau_\varepsilon(t)|^q(|\nabla u_\varepsilon(t)| + |\operatorname{div} u_\varepsilon(t)|)dx + \|\nabla g_\varepsilon(t)\|_{\mathbb{R}^n,q}\|\nabla \tau_\varepsilon(t)\|^{q-1}_{\mathbb{R}^n,q}.
\end{aligned}$$

Applying Sobolev's embedding theorem, dividing the result by $\|\partial_m \tau_\varepsilon(t)\|^{q-1}_q$, and taking the sum over m, $m = 1, \dots, n$, we deduce that

$$\frac{d}{dt}\|\nabla \tau_\varepsilon(t)\|_{\mathbb{R}^n,q} \leq C\big(\|\nabla g_\varepsilon(t)\|_{\mathbb{R}^n,q} + \|u_\varepsilon(t)\|_{H^2_q(\mathbb{R}^n)}\|\nabla \tau_\varepsilon(t)\|_{\mathbb{R}^n,q}\big),$$

due to (A.10). By Gronwall's Lemma, we obtain

$$\|\nabla \tau_\varepsilon\|_{T,\mathbb{R}^n,\infty,q} \leq C\big(\|\nabla \tau_{0,\varepsilon}\|_q + \|\nabla g_\varepsilon\|_{T,\mathbb{R}^n,1,q}\big)e^{C\|u_\varepsilon\|_{L_1(0,T;H^2_q(\mathbb{R}^n))}}.$$

Adding this equation to (A.4) and combining the result with

$$\|u_\varepsilon\|_{L_1(0,T;H_q^2(\mathbb{R}^n))} \le T^{\frac{p-1}{p}} \|u_\varepsilon\|_{L_p(0,T;H_q^2(\mathbb{R}^n))},$$

it follows the estimate

$$\|\tau_\varepsilon\|_{L_\infty(0,T;H_q^1(\mathbb{R}^n))} \le C\big(\|\tau_{0,\varepsilon}\|_{H_q^1(\mathbb{R}^n)} + \|g_\varepsilon\|_{L_1(0,T;H_q^1(\mathbb{R}^n))}\big) e^{CT^{\frac{p-1}{p}}\|u_\varepsilon\|_{L_p(0,T;H_q^2(\mathbb{R}^n))}}. \tag{A.11}$$

Moreover, by (A.5) and Sobolev's embedding theorem, we obtain

$$\begin{aligned} \|\partial_t \tau_\varepsilon\|_{T,\mathbb{R}^n,r,q} &\le \|g_\varepsilon\|_{T,\mathbb{R}^n,r,q} + \|u_\varepsilon.\nabla\tau_\varepsilon\|_{T,\mathbb{R}^n,r,q} \\ &\le \|g_\varepsilon\|_{T,\mathbb{R}^n,r,q} + \|u_\varepsilon\|_{T,\mathbb{R}^n,r,\infty}\|\tau_\varepsilon\|_{L_\infty(0,T;H_q^1(\mathbb{R}^n))}. \end{aligned} \tag{A.12}$$

By (A.11) and (A.12), the set $(\tau_\varepsilon)_{0<\varepsilon<1}$ is bounded in $L_\infty(0,T;H_q^1(\mathbb{R}^n))$ and the set $(\partial_t\tau_\varepsilon)_{0<\varepsilon<1}$ is bounded in $L_r(0,T;L_q(\mathbb{R}^n))$. Hence, there exists a sequence $(\varepsilon_m)_{m\in\mathbb{N}} \subset (0,1)$, with $\varepsilon_m \to 0$ for $m\to\infty$, and there exist functions $\tau \in L_\infty(0,T;H_q^1(\mathbb{R}^n))$ and $\tau_t \in L_r(0,T;L_q(\mathbb{R}^n))$, such that

$$\tau_{\varepsilon_m} \overset{*}{\rightharpoonup} \tau \quad \text{in} \quad L_\infty(0,T;H_q^1(\mathbb{R}^n)) \quad \text{and} \quad \partial_t\tau_{\varepsilon_m} \overset{*}{\rightharpoonup} \tau_t \quad \text{in} \quad L_r(0,T;L_q(\mathbb{R}^n)). \tag{A.13}$$

It holds the equality $\partial_t\tau = \tau_t$, since

$$\begin{aligned} (\tau_t|\varphi)_{T,\mathbb{R}^n} &= \lim_{m\to\infty}(\partial_t\tau_{\varepsilon_m}|\varphi)_{T,\mathbb{R}^n} = -\lim_{m\to\infty}(\tau_{\varepsilon_m}|\partial_t\varphi)_{T,\mathbb{R}^n} = -(\tau|\partial_t\varphi)_{T,\mathbb{R}^n} \\ &= (\partial_t\tau|\varphi)_{T,\mathbb{R}^n}, \quad \varphi \in C_c^\infty((0,T)\times\mathbb{R}^n). \end{aligned}$$

Next, we show that $(\tau_{\varepsilon_m})_{m\in\mathbb{N}}$ also converge in a strong sense. For $l,m\in\mathbb{N}$ we define

$$\tau_{l,m} := \tau_{\varepsilon_l} - \tau_{\varepsilon_m}.$$

This difference fulfills the equation

$$\begin{cases} \partial_t\tau_{l,m} + u_{\varepsilon_l}.\nabla\tau_{l,m} = g_{\varepsilon_l} - g_{\varepsilon_m} - (u_{\varepsilon_l} - u_{\varepsilon_m}).\nabla\tau_{\varepsilon_m} & \text{in } (0,T)\times\mathbb{R}^n, \\ \tau_{l,m}(0) = \tau_{0,\varepsilon_l} - \tau_{0,\varepsilon_m} & \text{in } \mathbb{R}^n. \end{cases}$$

For a solution of this equation, we already proved the a-priori estimate (A.4). Since $(u_{\varepsilon_l})_{l\in\mathbb{N}}$ is bounded in $L_1(0,T;H_q^2(\mathbb{R}^n))$, we infer

$$\begin{aligned} &\|\tau_{l,m}\|_{T,\mathbb{R}^n,\infty,q} \\ &\quad \le C\big(\|\tau_{0,\varepsilon_l} - \tau_{0,\varepsilon_m}\|_{\mathbb{R}^n,q} + \|g_{\varepsilon_l} - g_{\varepsilon_m}\|_{T,\mathbb{R}^n,1,q} + \|(u_{\varepsilon_l} - u_{\varepsilon_m}).\nabla\tau_{\varepsilon_m}\|_{T,\mathbb{R}^n,1,q}\big) \\ &\quad \le C\big(\|\tau_{0,\varepsilon_l} - \tau_{0,\varepsilon_m}\|_{\mathbb{R}^n,q} + \|g_{\varepsilon_l} - g_{\varepsilon_m}\|_{T,\mathbb{R}^n,1,q} + \|(u_{\varepsilon_l} - u_{\varepsilon_m})\|_{L_1(0,T;H_q^1(\mathbb{R}^n))}\|\tau_{\varepsilon_m}\|_{L_\infty(0,T;H_q^1(\mathbb{R}^n))}\big). \end{aligned}$$

Hence, the sequence $(\tau_{\varepsilon_m})_{m\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T],L_q(\mathbb{R}^n))$, since $(\tau_{\varepsilon_m})_{m\in\mathbb{N}}$ is bounded in $L_\infty(0,T;H_q^1(\mathbb{R}^n))$. Therefore, we have the convergence $\tau_{\varepsilon_n} \to \tau$ in $C([0,T],L_q(\mathbb{R}^n))$. In particular, we proved $\tau(0) = \tau_0$. In the last step, we show that τ satisfies (1.12) in the case that $\Omega = \mathbb{R}^n$. We test equation (A.5) with $\varphi \in C_c^\infty((0,T)\times\mathbb{R}^n)$

$$(\partial_t\tau_{\varepsilon_m}|\varphi)_{T,\mathbb{R}^n} + (u_{\varepsilon_m}.\nabla\tau_{\varepsilon_m}|\varphi)_{T,\mathbb{R}^n} = (g_{\varepsilon_m}|\varphi)_{T,\mathbb{R}^n}.$$

We take the limit in each term of this equation. By (A.13), we have the convergences

$$(\partial_t \tau_{\varepsilon_m}|\varphi)_{T,\mathbb{R}^n} \to (\partial_t \tau|\varphi)_{T,\mathbb{R}^n} \quad \text{and} \quad (g_{\varepsilon_m}|\varphi)_{T,\mathbb{R}^n} \to (g|\varphi)_{T,\mathbb{R}^n}, \quad m \to \infty.$$

Further, on account of the strong convergence $u_{\varepsilon_m} \to u$ in $L_1(0,T;L_{q'}(\mathbb{R}^n))$, the boundedness of $(\tau_\varepsilon)_{0<\varepsilon<1}$ in $L_\infty(0,T;H^1_q(\mathbb{R}^n))$, as well as the embedding $L_p(0,T;H^2_q(\mathbb{R}^n)) \to L_1(0,T;L_{q'}(\mathbb{R}^n))$, with $\frac{1}{q}+\frac{1}{q'}=1$, we infer

$$\begin{aligned}
&|(u_{\varepsilon_m}.\nabla\tau_{\varepsilon_m} - u.\nabla\tau|\varphi)_{T,\mathbb{R}^n}| \\
&\quad \le \sum_{j,k,l=1}^{n} |((u_{\varepsilon_m,j}-u_j)\partial_j\tau_{\varepsilon_m,k,l}|\varphi_{k,l})_{T,\mathbb{R}^n}| + |(u_j\partial_j(\tau_{\varepsilon_m,k,l}-\tau_{k,l})|\varphi_{k,l})_{T,\mathbb{R}^n}| \\
&\quad \le \|u_{\varepsilon_m}-u\|_{T,\mathbb{R}^n,1,q'}\|\|\nabla\tau_{\varepsilon_m}\|_{T,\mathbb{R}^n,\infty,q}\|\varphi\|_{T,\Omega,\infty,\infty} + \sum_{j,k,l=1}^{n} |(\partial_j(\tau_{\varepsilon_m,k,l}-\tau_{k,l})|u_j\varphi_{k,l})_{T,\mathbb{R}^n}| \\
&\qquad \to 0, \quad m \to \infty.
\end{aligned}$$

Thus, τ is the solution of (1.12) in the case that $\Omega = \mathbb{R}^n$. By lower semi continuity of the norm with respect to the weak*-convergence, the a-priori estimates for τ is also valid. The a-priori estimate for $\partial_t\tau$ follows from (1.12) by Hölder's inequality.

In the case that $\Omega \subset \mathbb{R}^n$ is a domain with a uniform C^2-boundary, we extend the date u, g and τ_0 with a suitable continuous extension operator $\mathcal{E}_\Omega$ (for the existence of such an operator, we refer to Adams and Fournier [AF03, Theorem 5.24]) and solve the problem

$$\left\{\begin{array}{rcll} \partial_t\tau_{\mathcal{E}} + (\mathcal{E}_\Omega u).\nabla\tau_{\mathcal{E}} &=& \mathcal{E}_\Omega g & \text{in } (0,T)\times\mathbb{R}^n, \\ \tau_{\mathcal{E}}(0) &=& \mathcal{E}_\Omega\tau_0 & \text{in } \mathbb{R}^n. \end{array}\right.$$

The function $\tau = \tau_{\mathcal{E}}|_\Omega$ now solves the original problem (1.12) and according to the continuity of the extension operator, the a-priori estimates hold for τ. The estimate of $\partial_t\tau$ follows from (1.12) by Hölder's inequality. $\square$

Appendix B

Function spaces and Nemytskij operators

We give a proof of the propositions on Nemytskij operators (Proposition 1.17) and Proposition 1.19. A result similarly to Proposition 1.17 was established by Runst and Sickel [RS96, Theorem 5.5.3.1] and we use similar arguments in the proof.

Proof of Proposition 1.17. Let $R_0 > 0$ and $\Psi \in C^1(\mathbb{R}^N)$. Let us first prove the estimate

$$\|\Psi(f)\|_{\mathbb{H}_u^\infty(T,\Gamma)} \le C(R_0, \Psi), \quad f \in B_{\mathbb{H}_u^\infty(T,\Gamma)}(0, R_0), \tag{B.1}$$

with a constant C, which is independent of T, $0 < T < T_0$. We have

$$\|\Psi(f)\|_{T,\Gamma,\infty,\infty} \le \|\Psi\|_{B(0,R_0),\infty}, \quad f \in B_{\mathbb{H}_u^\infty(T,\Gamma)}(0, R_0),$$

and, by the mean value theorem, we infer

$$\begin{aligned}
&[\Psi(f)]_{\mathbb{H}_u(T,\Gamma)} \\
&\quad = \Big(\int_0^T\int_0^T \frac{\|\Psi(f)(t)-\Psi(f)(s)\|_{p,\Gamma}^p}{|t-s|^{\frac12+\frac p2}}dsdt\Big)^{\frac1p} + \Big(\int_0^T\int_\Gamma\int_\Gamma \frac{|\Psi(f)(t,x)-\Psi(f)(t,y)|^p}{|x-y|^{n-2+p}}dxdydt\Big)^{\frac1p} \\
&\quad \le \|\Psi\|_{W^1_\infty(B(0,R_0))}\times \\
&\qquad \times\Big(\Big(\int_0^T\int_0^T \frac{\|f(t)-f(s)\|_{p,\Gamma}^p}{|t-s|^{\frac12+\frac p2}}dsdt\Big)^{\frac1p} + \Big(\int_0^T\int_\Gamma\int_\Gamma \frac{|f(t,x)-f(t,y)|^p}{|x-y|^{n-2+p}}dxdydt\Big)^{\frac1p}\Big) \\
&\quad \le \|\Psi\|_{W^1_\infty(B(0,R_0))}[f]_{\mathbb{H}_u(T,\Gamma)}, \quad f \in B_{\mathbb{H}_u^\infty(T,\Gamma)}(0,R_0),
\end{aligned}$$

which proves (B.1).

Let again $R_0 > 0$ and $\Psi \in C^1(\mathbb{R}^N)$. We prove the estimate

$$\|\Psi(f) - \Psi(0)\|_{L_\infty(0,T;H_p^1(\Omega))} \le C(R_0, \Psi), \quad f \in B_{L_\infty(0,T;H_p^1(\Omega))}(0, R_0), \tag{B.2}$$

with a constant C, which is independent of T, $0 < T < T_0$. Due to Sobolev's embedding theorem, there exists a constant C_0 with

$$\|f\|_{T,\Omega,\infty,\infty} \le C_0(R_0), \quad f \in B_{L_\infty(0,T;H_p^1(\Omega))}(0, R_0).$$

By the mean value theorem, we deduce that

$$\|\Psi(f)-\Psi(0)\|_{T,\Omega,\infty,p} \leq \|\Psi\|_{W^1_\infty(B(0,C_0(R_0)))}\|f\|_{T,\Omega,\infty,p} \leq C(\Psi,R_0), \quad f\in B_{L_\infty(0,T;H^1_p(\Omega))}(0,R_0),$$

and, applying the chain rule

$$\begin{aligned}\|\nabla(\Psi(f)-\Psi(0))\|_{T,\Omega,\infty,p} &\leq \|\Psi\|_{W^1_\infty(B(0,C_0(R_0)))}\|\nabla f\|_{L_\infty(0,T;H^1_p(\Omega))}\\ &\leq C(\Psi,R_0), \quad f\in B_{L_\infty(0,T;H^1_p(\Omega))}(0,R_0),\end{aligned}$$

which establishes (B.2).

Our next subject is the Fréchet differentiability of the Nemytskij operators. In a first step, we consider a general situation. Let $d\in\mathbb{N}$, $G\subset\mathbb{R}^d$, $X\subset\{f\colon G\to\mathbb{R}\}$ be a function space and $Y_0\subset\{\tilde\Psi\in C(\mathbb{R}^N)\colon \tilde\Psi(0)=0\}$, satisfying the following assumptions:

(FD1) X is a Banach algebra.

(FD2) For every $\tilde\Psi\in Y_0$ and $f\in X$, it holds $\tilde\Psi(f)\in X$.

(FD3) For $\tilde\Psi\in Y_0$ and $R_0>0$, there exists a constant $C(\tilde\Psi,R_0)$ with

$$\|\tilde\Psi(f)\|_X\leq C(\tilde\Psi,R_0), \quad f\in B_X(0,R_0).$$

Fix $\Psi_0\in C^1(\mathbb{R}^N)$ with $\Psi_0,\Psi_0'-\Psi_0'(0)\in Y_0$. We show the continuity of the Nemytskij operator $\Psi_0\colon X\to X$. Let $f,h\in X$ with $\|h\|_X\leq 1$. By Taylor's formula, it follows that

$$\Psi_0(f+h)(x)-\Psi_0(f)(x)=\int_0^1\Psi_0'(f+sh)(x)h(x)ds, \quad x\in G.$$

Applying **(FD1)**, **(FD2)**, and **(FD3)**, we infer

$$\begin{aligned}\Big\|\Psi_0(f+h)-\Psi_0(f)\Big\|_X &\leq \Big\|\int_0^1\Psi_0'(f+sh)hds\Big\|_X\\ &\leq \Big\|\int_0^1(\Psi_0'(f+sh)-\Psi_0'(0))hds\Big\|_X+C\Big\|h\Big\|_X\\ &\leq C(\sup_{s\in(0,1)}\|(\Psi_0'(f+sh)-\Psi_0'(0))\|_X\|h\|_X+\|h\|_X)\\ &\leq C\|h\|_X.\end{aligned} \tag{B.3}$$

This implies the continuity of $\Psi_0\colon X\to X$.

Next, we fix $\Psi_0\in C^2(\mathbb{R}^N)$ with $\Psi_0,\Psi_0'-\Psi_0'(0),\Psi_0''-\Psi_0''(0)\in Y_0$. We prove, that the Nemytskij operator $\Psi_0\colon X\to X$ is continuously Fréchet differentiable. Let $f,h\in X$ with $\|h\|_X\leq 1$. By Taylor's formula, it follows that

$$\Psi_0(f+h)(x)-\Psi_0(f)(x)-\Psi_0'(f)(x)h(x)=\int_0^1(1-s)h(x)^T\Psi_0''(f+sh)(x)h(x)ds, \quad x\in G.$$

Applying **(FD1)**, **(FD2)**, and **(FD3)**, we estimate the integral on the right-hand side similar to above

$$\begin{aligned}\Big\|\int_0^1(1-s)h^T\Psi_0''(f+sh)hds\Big\|_X &\le \Big\|\int_0^1(1-s)h^T(\Psi_0''-\Psi_0''(0))(f+sh)hds\Big\|_X + C\Big\|h\Big\|_X^2\\ &\le C(\sup_{s\in(0,1)}\|h^T(\Psi_0''-\Psi_0''(0))(f+sh)h\|_X+\|h\|_X^2)\\ &\le C\|h\|_X^2.\end{aligned}$$

This shows that $\Psi_0\colon X\to X$ is Fréchet differentiable with Fréchet derivative $D\Psi_0(f)[h]=\Psi_0'(f)h$. Now, we show that the Fréchet derivative $D\Psi_0\colon X\to\mathcal{L}(X)$ is continuous. We already showed that the Nemytskij $\Psi_0'-\Psi_0'(0)\colon X\to X$ is continuous. Applying (B.3), we deduce for $f,g\in X$ with $\|g\|_X\le 1$ that

$$\begin{aligned}\|D\Psi_0(f+g)-D\Psi_0(f)\|_{\mathcal{L}(X)} &\le \sup_{\|h\|_X\le 1}\|(\Psi_0'(f+g)-\Psi_0'(f))h\|_X\\ &\le \sup_{\|h\|_X\le 1}\|((\Psi_0'-\Psi_0'(0))(f+g)-(\Psi_0'-\Psi_0'(0))(f))h\|_X\\ &\le C\|(\Psi_0'-\Psi_0'(0))(f+g)-(\Psi_0'-\Psi_0'(0))(f)\|_X\le C\|g\|_X.\end{aligned}$$

This implies the continuity of $D\Psi_0\colon X\to\mathcal{L}(X)$.

Summarized, we proved that for $\Psi_0\in C^2(\mathbb{R}^N)$ with $\Psi_0,\Psi_0'-\Psi_0'(0),\Psi_0''-\Psi_0''(0)\in Y_0$ the corresponding Nemytskij operator $\Psi_0\colon X\to X$ is continuously Fréchet differentiable.

Next, we apply the abstract result to following concrete situations:

(C1) $(X,Y_0)=(W_p^s(\Gamma),\{\tilde\Psi\in C^1(\mathbb{R}^N)\colon\tilde\Psi(0)=0\})$ and $\Psi_0\in C^3(\mathbb{R}^N)\cap Y_0$, where $s\in\{1-\frac{3}{p},3-\frac{1}{p}\}$

(C2) $(X,Y_0)=(\mathbb{H}_u(T,\Gamma),\{\tilde\Psi\in C^1(\mathbb{R}^N)\colon\tilde\Psi(0)=0\})$ and $\Psi_0\in C^3(\mathbb{R}^N)\cap Y_0$,

(C3) $(X,Y_0)=(BUC([0,T],BUC(\overline\Omega)),\{\tilde\Psi\in C(\mathbb{R}^N)\colon\tilde\Psi(0)=0\})$ and $\Psi_0\in C^2(\mathbb{R}^N)\cap Y_0$,

(C4) $(X,Y_0)=(BUC([0,T],BUC(\dot{\mathbb{R}}^n)),\{\tilde\Psi\in C(\mathbb{R}^N)\colon\tilde\Psi(0)=0\})$ and $\Psi_0\in C^2(\mathbb{R}^N)\cap Y_0$,

(C5) $(X,Y_0)=(L_\infty(0,T;H_p^1(\Omega)),\{\tilde\Psi\in C^1(\mathbb{R}^N)\colon\tilde\Psi(0)=0\})$ and $\Psi_0\in C^3(\mathbb{R}^N)\cap Y_0$.

We prove that the Nemytskij operator $\Psi_0\colon X\to X$ is continuously Fréchet differentiable. It is sufficient to verify **(FD2)** and **(FD3)**, since the Banach spaces $W_p^s(\Gamma)$, $\mathbb{H}_u(T,\Gamma)$, $BUC([0,T],BUC(\overline\Omega))$, $BUC([0,T],BUC(\dot{\mathbb{R}}^n))$, and $L_\infty(0,T;H_p^1(\Omega))$ are Banach algebras (see Proposition 1.16).

(C1): Let $R_0>0$, $\tilde\Psi\in C^1(\mathbb{R}^N)$ with $\tilde\Psi(0)=0$, and $f\in W_p^s(\Gamma)$ with $\|f\|_{W_p^s(\Gamma)}\le R_0$. By the Sobolev embedding theorem, there exists a constant C_0 with

$$\|f\|_{\Gamma,\infty}\le C_0(R_0).$$

By the mean value theorem and $\tilde\Psi(0)=0$, it follows that

$$\|\tilde\Psi(f)\|_{\Gamma,p}=\|\tilde\Psi(f)-\tilde\Psi(0)\|_{\Gamma,p}\le\|\tilde\Psi\|_{W_\infty^1(B(0,C_0(R_0)))}\|f\|_{\Gamma,p}\le C(R_0,\tilde\Psi).$$

Further, we deduce that

$$\begin{aligned}[\tilde\Psi(f)]_{W_p^s(\Gamma)} &= \Big(\int_\Gamma\int_\Gamma\frac{|\tilde\Psi(f(x))-\tilde\Psi(f(y))|^p}{|x-y|^{n-1+sp}}\Big)^{\frac{1}{p}}\le\|\tilde\Psi\|_{W_\infty^1(B(0,C_0(R_0)))}\Big(\int_\Gamma\int_\Gamma\frac{|f(x)-f(y)|^p}{|x-y|^{n-1+sp}}\Big)^{\frac{1}{p}}\\ &\le C(R_0,\tilde\Psi).\end{aligned}$$

This proves, that the Nemyskij operator $\Psi_0 \colon W_p^s(\Gamma) \to W_p^s(\Gamma)$ is continuously Fréchet differentiable.
(C2): Let $R_0 > 0$, $\tilde{\Psi} \in C^1(\mathbb{R}^N)$ with $\tilde{\Psi}(0) = 0$, and $f \in \mathbb{H}_u(T,\Gamma)$ with $\|f\|_{\mathbb{H}_u(T,\Gamma)} \leq R_0$. By the proposition on embedding theorems (Proposition 1.14), there exists a constant C_0 (this constant depends in general on T) with

$$\|f\|_{T,\Gamma,\infty,\infty} \leq C_0(R_0).$$

By the mean value theorem and $\tilde{\Psi}(0) = 0$, we obtain

$$\|\tilde{\Psi}(f)\|_{T,\Gamma,p,p} = \|\tilde{\Psi}(f) - \tilde{\Psi}(0)\|_{T,\Gamma,p,p} \leq \|\tilde{\Psi}\|_{W^1_\infty(B(0,C_0(R_0)))}\|f\|_{T,\Gamma,p,p}.$$

Combining the last inequality with (B.1), we deduce that

$$\|\tilde{\Psi}(f)\|_{\mathbb{H}_u(T,\Gamma)} \leq C(\Psi, R_0).$$

This establishes, that the Nemyskij operator $\Psi_0 \colon \mathbb{H}_u(T,\Gamma) \to \mathbb{H}_u(T,\Gamma)$ is continuously Fréchet differentiable.
(C3): For R_0 and $\tilde{\Psi} \in C(\mathbb{R}^N)$, it follows

$$\|\tilde{\Psi}(f)\|_{T,\Omega,\infty,\infty} \leq \|\tilde{\Psi}\|_{B(0,R_0),\infty}, \quad f \in B_{BUC([0,T],BUC(\overline{\Omega}))}(0,R_0).$$

This proves, that the Nemyskij operator $\Psi_0 \colon BUC([0,T], BUC(\overline{\Omega})) \to BUC([0,T], BUC(\overline{\Omega}))$ is continuously Fréchet differentiable.
(C4): This follows the same way as **(C3)**.
(C5): Estimate (B.2) implies that the Nemyskij operator

$$\Psi_0 \colon L_\infty(0,T; H^1_p(\Omega)) \to L_\infty(0,T; H^1_p(\Omega))$$

is continuously Fréchet differentiable.

By now, we investigated the Fréchet differentiable of Nemytskij operators with $\Psi_0(0) = 0$. For a function with $\Psi(0) \neq 0$, we can apply the proven results on $\Psi - \Psi(0)$. Let $\Psi \in C^2(\mathbb{R}^N)$. Since the constant operator

$$\Psi(0) \colon X \to X, \quad X \in \{BUC([0,T], BUC(\overline{\Omega})), BUC([0,T], BUC(\dot{\mathbb{R}}^n))\}$$

is smooth, we have the Fréchet differentiability of

$$\Psi \colon X \to X, \quad X \in \{BUC([0,T], BUC(\overline{\Omega})), BUC([0,T], BUC(\dot{\mathbb{R}}^n))\}.$$

Let from now on $\Psi \in C^3(\mathbb{R}^N)$. Due to the continuous embeddings $\mathbb{H}_u(T,\Gamma) \to \mathbb{H}_u^\infty(T,\Gamma)$ and $W_p^s(\Gamma) \to \widehat{W}_p^s(\Gamma) \cap L_\infty(\Gamma)$, we infer the Fréchet differentiability of

$$\Psi - \Psi(0) \colon \mathbb{H}_u(T,\Gamma) \to \mathbb{H}_u^\infty(T,\Gamma) \quad \text{and} \quad \Psi - \Psi(0) \colon W_p^s(\Gamma) \to \widehat{W}_p^s(\Gamma) \cap L_\infty(\Gamma).$$

Since the constant operator

$$\Psi(0) \colon \mathbb{H}_u(T,\Gamma) \to \mathbb{H}_u^\infty(T,\Gamma) \quad \text{and} \quad \Psi - \Psi(0) \colon W_p^s(\Gamma) \to \widehat{W}_p^s(\Gamma) \cap L_\infty(\Gamma)$$

is smooth, we obtain the assertion. □

Proof of Proposition 1.19. The first estimate in Proposition 1.19 follow by the mean value theorem and the previous proposition on Nemytskij operators (Proposition 1.17). In the proof of the previous proposition (see **(C2)**), we showed that the Nemytskij operator

$$(\Psi_2 - \Psi_2(0))\colon \mathbb{H}_u(T,\Gamma) \to \mathbb{H}_u(T,\Gamma)$$

is continuously Fréchet differentiable. Since $\mathbb{H}_u(T,\Gamma) \hookrightarrow L_\infty(0,T;L_\infty(\Gamma))$, this implies that the Nemytskij operator

$$(\Psi_2 - \Psi_2(0))\colon \mathbb{H}_u(T,\Gamma) \cap L_\infty(0,T;L_\infty(\Gamma)) \to \mathbb{H}_u(T,\Gamma) \cap L_\infty(0,T,L_\infty(\Gamma))$$

is continuously Fréchet differentiable. Let now $R_0 > 0$ and $f, g \in B_{\mathbb{H}_u(T,\Gamma)\cap L_\infty(0,T;L_\infty(\Gamma))}(0,R_0)$ with $f(0) = g(0)$. By the mean value theorem, it follows that

$$\begin{aligned} &\|\Psi_2(f) - \Psi_2(g)\|_{{}_0\mathbb{H}_u(T,\Gamma)} \\ &\qquad \le C \sup_{\|h\|_{\mathbb{H}_u(T,\Gamma)} + \|h\|_{T,\Gamma,\infty,\infty} \le R_0} \|D\Psi_2(h)\|_{\mathcal{L}(\mathbb{H}_u(T,\Gamma)\cap L_\infty(0,T;L_\infty(\Gamma)))} \|f-g\|_{{}_0\mathbb{H}_u(T,\Gamma)}. \end{aligned}$$

Further, by the previous proposition on Nemytskij operators and the proposition on pointwise multiplications, we estimate

$$\begin{aligned} \|D\Psi_2(h)\|_{\mathcal{L}(\mathbb{H}_u(T,\Gamma)\cap L_\infty(0,T;L_\infty(\Gamma)))} &= \sup_{\|z\|_{\mathbb{H}_u(T,\Gamma)\cap L_\infty(0,T;L_\infty(\Gamma))} \le 1} \|D\Psi(h)z\|_{\mathbb{H}_u(T,\Gamma)\cap L_\infty(0,T;L_\infty(\Gamma))} \\ &= \sup_{\|z\|_{\mathbb{H}_u(T,\Gamma)\cap L_\infty(0,T;L_\infty(\Gamma))} \le 1} \|\Psi'(h)z\|_{\mathbb{H}_u(T,\Gamma)\cap L_\infty(0,T;L_\infty(\Gamma))} \\ &\le C\|\Psi'(h)\|_{\mathbb{H}_u^\infty(T,\Gamma)}, \\ &\le C(\Psi, R_0), \quad h \in B_{\mathbb{H}_u(T,\Gamma)\cap L_\infty(0,T;L_\infty(\Gamma))}(0,R_0). \end{aligned}$$

Hence, we established the second estimate in Proposition 1.19.

Next, we prove the third estimate in Proposition 1.19. Let first $G \in \{\Gamma, (0,T)\}$, $0 < s < 1$, $\tilde{f}, \tilde{g} \in W_p^s(G)$ and $\tilde{\Psi}_3 \in BUC^3(\mathbb{R})$. Taylor's theorem yields

$$\tilde{\Psi}_3(\xi_2) - \tilde{\Psi}_3(\xi_1) - \tilde{\Psi}_3'(\xi_1)(\xi_2-\xi_1) = \int_0^1 (1-t)(\xi_2-\xi_1)^2 \tilde{\Psi}_3''((1-t)\xi_1 + t\xi_2)dt, \quad \xi_1, \xi_2 \in \mathbb{R}. \tag{B.4}$$

We introduce $\tilde{h} = \tilde{f} - \tilde{g}$,

$$a_z(t) = t\tilde{f}(z) + (1-t)\tilde{g}(z) = t\tilde{h}(z) + \tilde{g}(z), \quad z \in G, \quad t \in (0,1),$$

and

$$a(t,s) = sa_x(t) + (1-s)a_y(t), \quad x, y \in G, \quad s, t \in (0,1).$$

Let $x, y \in G$. We have

$$\tilde{\Psi}_3(\tilde{f}(x)) - \tilde{\Psi}_3(\tilde{g}(x)) - \tilde{\Psi}_3'(\tilde{g}(x))\tilde{h}(x) = \int_0^1 (1-t)\tilde{h}(x)^2 \tilde{\Psi}_3''(a_x(t))dt, \tag{B.5}$$

$$\tilde{\Psi}_3(\tilde{f}(y)) - \tilde{\Psi}_3(\tilde{g}(y)) - \tilde{\Psi}_3'(\tilde{g}(y))\tilde{h}(y) = \int_0^1 (1-t)\tilde{h}(y)^2 \tilde{\Psi}_3''(a_y(t))dt. \tag{B.6}$$

On account of (B.5) and (B.6), we obtain due to Taylor's theorem

$$\begin{aligned}
&\big(\tilde\Psi_3(\tilde f(x)) - \tilde\Psi_3(\tilde g(x)) - \tilde\Psi_3'(\tilde g(x))\tilde h(x)\big) - \big(\tilde\Psi_3(\tilde f(y)) - \tilde\Psi_3(\tilde g(y)) - \tilde\Psi_3'(\tilde g(y))\tilde h(y)\big)\\
&\quad = \int_0^1 (1-t)\big(\tilde h(x)^2\tilde\Psi_3''(a_x(t)) - \tilde h(y)^2\tilde\Psi_3''(a_y(t))\big)dt\\
&\quad = \int_0^1 (1-t)\big((\tilde h(x)-\tilde h(y))(\tilde h(x)+\tilde h(y))\tilde\Psi_3''(a_x(t)) + \tilde h(y)^2(\tilde\Psi_3''(a_x(t)) - \tilde\Psi_3''(a_y(t)))\big)dt\\
&\quad = \int_0^1 (1-t)\big((\tilde h(x)-\tilde h(y))(\tilde h(x)+\tilde h(y))\tilde\Psi_3''(a_x(t))\\
&\qquad + \tilde h(y)^2\int_0^1 (a_x(t)-a_y(t))\tilde\Psi_3'''(a(t,s))ds\big)dt\\
&\quad = \int_0^1 (1-t)\big((\tilde h(x)-\tilde h(y))(\tilde h(x)+\tilde h(y))\tilde\Psi_3''(a_x(t))\\
&\qquad + \tilde h(y)^2\int_0^1 (t(\tilde h(x)-\tilde h(y)) + \tilde g(x)-\tilde g(y))\tilde\Psi_3'''(a(t,s))ds\big)dt.
\end{aligned}$$

By this identity, we deduce that

$$\begin{aligned}
&\big|\big(\tilde\Psi_3(\tilde f(x)) - \tilde\Psi_3(\tilde g(x)) - \tilde\Psi_3'(\tilde g(x))\tilde h(x)\big) - \big(\tilde\Psi_3(\tilde f(y)) - \tilde\Psi_3(\tilde g(y)) - \tilde\Psi_3'(\tilde g(y))\tilde h(y)\big)\big|\\
&\qquad\qquad \le \|\tilde\Psi_3\|_{W^3_\infty(\mathbb{R})}\big(2\|\tilde h\|_{G,\infty}|\tilde h(x)-\tilde h(y)| + \|\tilde h\|^2_{G,\infty}(|\tilde h(x)-\tilde h(y)| + |\tilde g(x)-\tilde g(y)|)\big),
\end{aligned}$$

and hence

$$\begin{aligned}
\text{(B.7)}\quad &\big[\tilde\Psi_3(\tilde f) - \tilde\Psi_3(\tilde g) - \tilde\Psi_3'(\tilde g)(\tilde f - \tilde g)\big]_{W^s_p(G)}\\
&\qquad\qquad \le \|\tilde\Psi_3\|_{W^3_\infty(\mathbb{R})}\big(2\|\tilde h\|_{G,\infty}[\tilde h]_{W^s_p(G)} + \|\tilde h\|^2_{G,\infty}([\tilde h]_{W^s_p(G)} + [\tilde g]_{W^s_p(G)})\big).
\end{aligned}$$

Let now $\Psi \in C^3(\mathbb{R})$ and $f, g \in B_{\mathbb{H}_u(T,\Gamma)\cap L_\infty(0,T;L_\infty(\Gamma))}(0, R_0)$ with $f - g \in {}_0\mathbb{H}_u(T,\Gamma)$, and $h := f - g$. By Taylor expansion (B.4) and the proposition on pointwise multiplication (Proposition 1.16), it follows that

$$\|\Psi_3(f) - \Psi_3(g) - \Psi_3'(g)(f-g)\|_{T,\Gamma,p,p} \le \|\Psi_3\|_{W^2_\infty(0,R_0)}\|h\|^2_{{}_0\mathbb{H}_u(T,\Gamma)} \le C\|h\|_{{}_0\mathbb{H}_u(T,\Gamma)}.$$

Further, we use (1.13), (B.7), and the proposition on pointwise multiplications to estimate the homogeneous norm:

$$\begin{aligned}
&[\Psi_3(f) - \Psi_3(g) - \Psi_3'(g)(f-g)]_{\mathbb{H}_u(T,\Gamma)}\\
&\quad = \Big(\int_\Gamma [(\Psi_3(f) - \Psi_3(g) - \Psi_3'(g)(f-g))(\cdot,x)]^p_{W_p^{\frac12-\frac1{2p}}(0,T)}dx\Big)^{\frac1p}\\
&\qquad + \Big(\int_0^T [(\Psi_3(f) - \Psi_3(g) - \Psi_3'(g)(f-g))(t,\cdot)]_{W_p^{1-\frac1p}(\Gamma)}dt\Big)^{\frac1p}\\
&\quad \le \|\Psi_3\|_{W^3_\infty(0,R_0)}\big(2\|h\|_{T,\Gamma,\infty,\infty}[h]_{{}_0\mathbb{H}_u(T,\Gamma)} + \|h\|^2_{T,\Gamma,\infty,\infty}([h]_{{}_0\mathbb{H}_u(T,\Gamma)} + [g]_{\mathbb{H}_u(T,\Gamma)})\big)\\
&\quad \le C\|h\|^2_{{}_0\mathbb{H}_u(T,\Gamma)}.
\end{aligned}$$

This proves the third estimate of Proposition 1.19. □

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